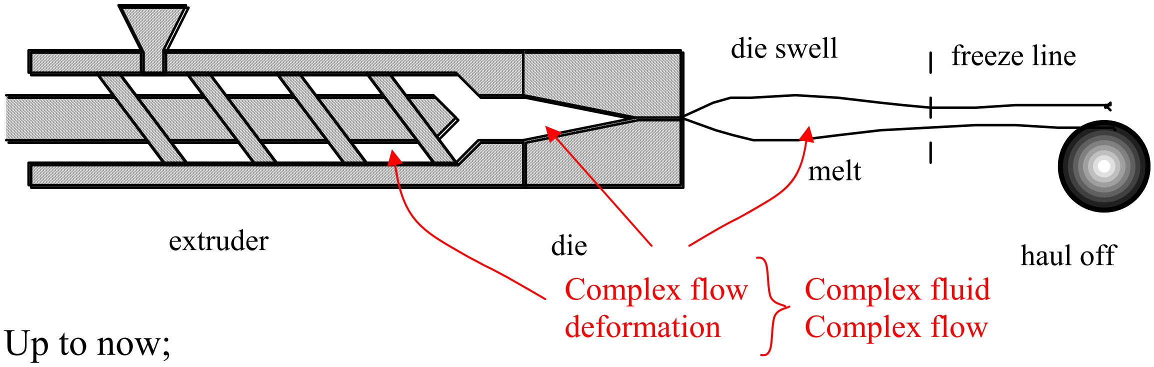
Section 4. Generalised deformations.

Leading to an ability to numerically simulate processing behaviour.

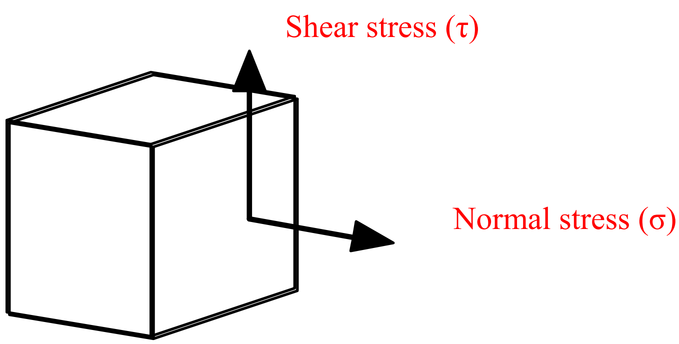
Objective. Ability to model complex viscoelastic processing conditions.

For example, polymer extrusion.

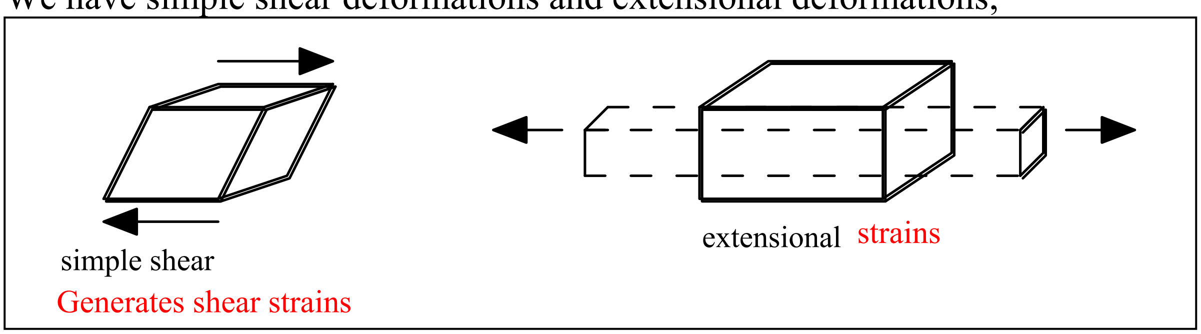


Up to now;
We have dealt only with shear stress τ and simple shear rate $\dot{\gamma}$
Life is more complex.

We have shear stresses and normal stresses;



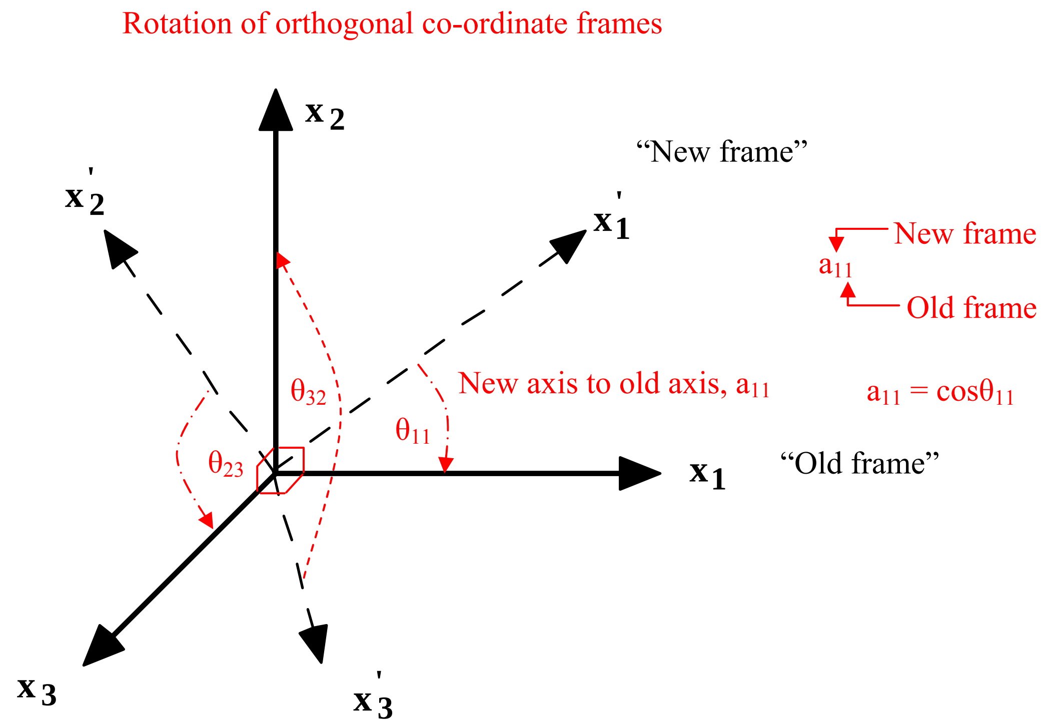
We have simple shear deformations and extensional deformations;



We need tools to handle generalised stress, strain rate and constitutive equations.
We need TENSORS.

1. Coordinate transformations. *A necessary starting step*

Use subscripts 1,2 and 3 to identify orthogonal axis.



x_1, x_2, x_3 "old" orthoganal coordinate frame.

x'_1, x'_2, x'_3 "New" orthoganal coordinate frame.

Direction cosine matrix;

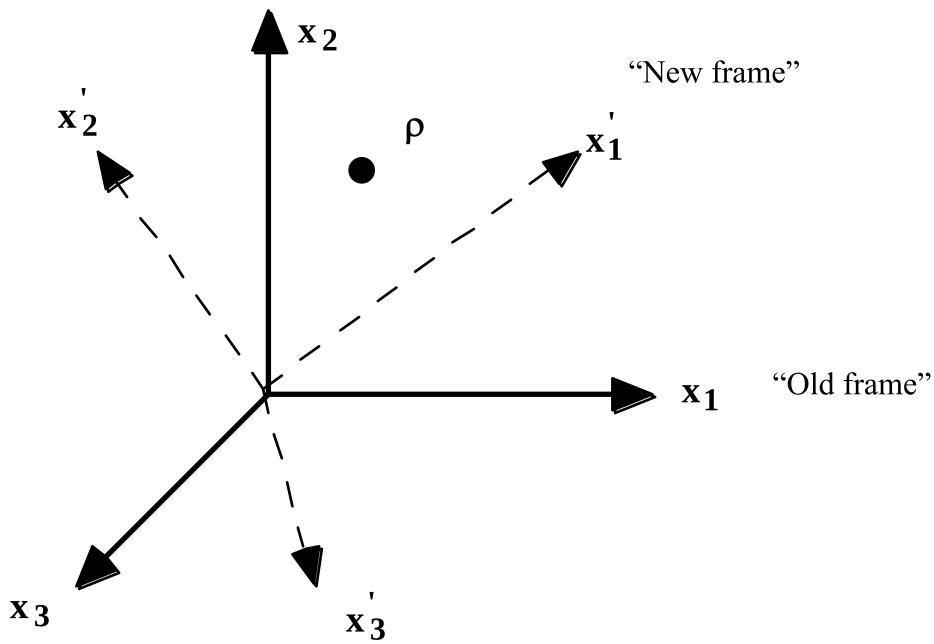
$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$ ***In general nine components***

$$a_{11} = \cos\theta_{11} \ , \quad a_{23} = \cos\theta_{23} \Longrightarrow$$ Defines direction cosine

2. Coordinate transformation of a scalar. *(boring but instructive)*

Scalars; ⟹ Directionally independent,

Temperature
Density
} Examples of scalar quantities

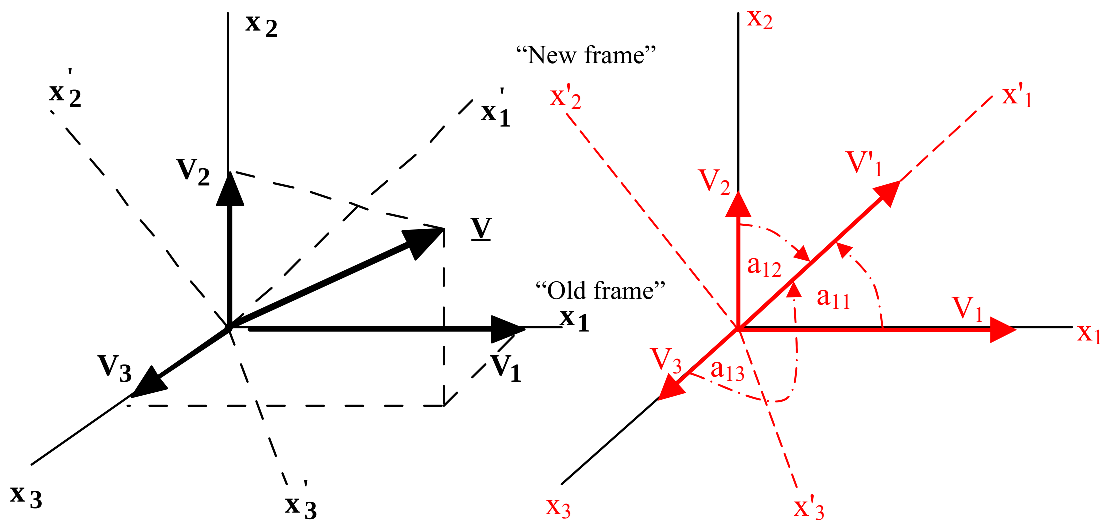


density ρ

$\rho' = \rho$ **ρ is independant of coordinate rotation** ⟹ Insensitive to co-ordinate system/translation

What about pressure? Scalar, vector or tensor?

3.Coordinate transformation of a vector *(instructive because we can see what is going on)*



Velocity component in new co-ordinate frame

$$V_1' = a_{11}V_1 + a_{12}V_2 + a_{13}V_3$$

Direction cosine, where $a_{11} = \cos\theta_{11}$

similarly,

$$V_2' = a_{21}V_1 + a_{22}V_2 + a_{23}V_3$$

$$V_3' = a_{31}V_1 + a_{32}V_2 + a_{33}V_3$$

In general,

$$V_i' = a_{i1}V_1 + a_{i2}V_2 + a_{i3}V_3$$

where i = 1, 2 or 3

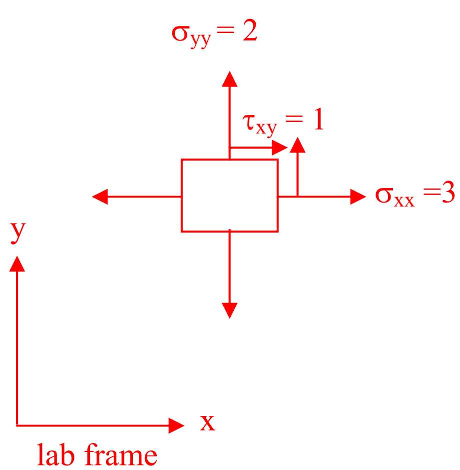
Einstein notation.

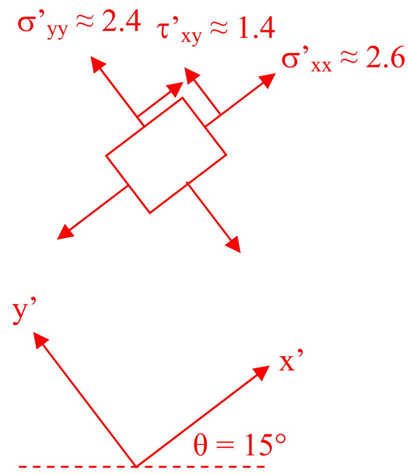
$$V_i' = \sum a_{ij}V_j \qquad j = 1, 2, \text{and } 3$$

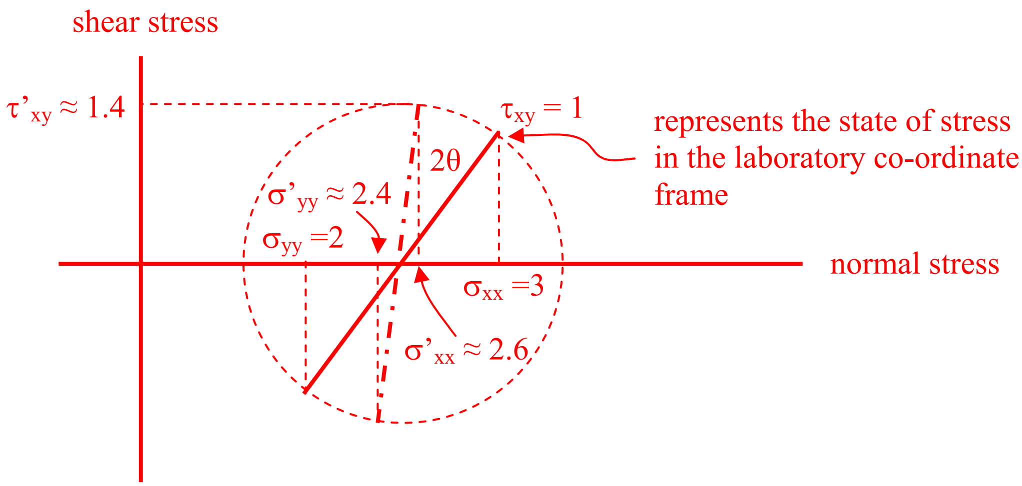
$$V_i' = a_{ij}V_j = \sum a_{ij}V_j$$ where j is dummy suffix notation meaning

$$V_i' = a_{i1}V_1 + a_{i2}V_2 + a_{i3}V_3$$

Intro tensor 2 subscript properties. Remembering Mohrs Circle?
(*not formally part of course but relevant* ***)***
Stress given in lab frame.
What are stress components in frame rotated clockwise by 15^0?
What and where are Principal stress components ?

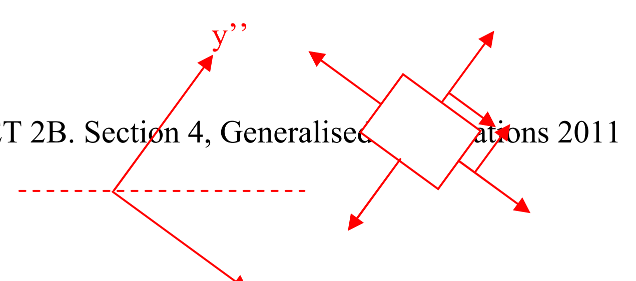






there exists one co-ordinate frame where $\tau_{xy} = 0$, which is when the orientation of the co-ordinate frame with respect to the laboratory frame is such that the state of stress line is superimposed on the normal stress axis

$$\begin{vmatrix} \sigma''_{xx} & 0 \\ 0 & \sigma''_{yy} \end{vmatrix}$$ Principal stress



In general 3D flow there are 6 stress components. If the stresses are rotated into the Principal frame this number is then reduced to 3 stresses

Double subscript property

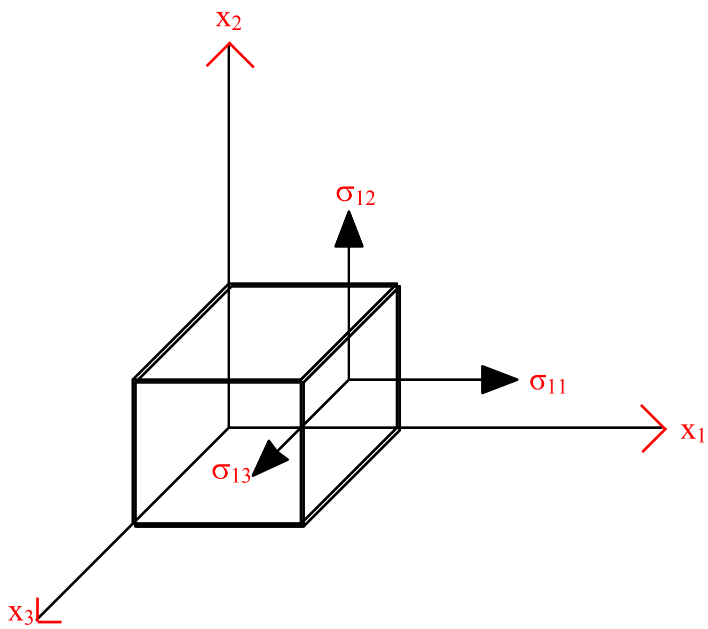
4. Coordinate transformation of a Tensor,

(Stress and strain rates are tensors, so this is important to us)

(say stress σ = (the real stress acting on body).

(Using σ's only, not σ and τ's, τ will take on new meaning later)

σ_{ij} **i = the face , j = the direction**



$$\begin{vmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ \sigma_{12} & \sigma_{22} & \sigma_{23} \\ \sigma_{13} & \sigma_{23} & \sigma_{33} \end{vmatrix}$$

Six components of stress

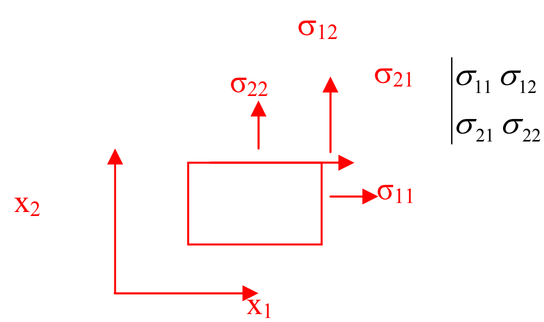
Example in 2D

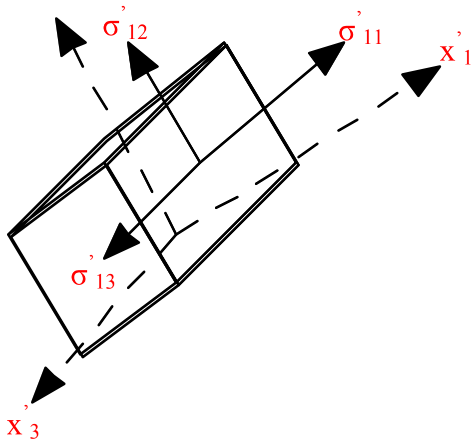
Matrix is symmetrical $\sigma_{ij} = \sigma_{ji}$

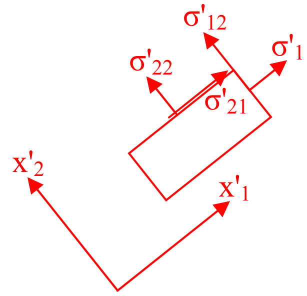
σ_{11} , σ_{22} , σ_{33} **Normal stresses**

σ_{12} , σ_{13} , σ_{23} **Shear stresses**

Transform to "new" coordinate frame.







$$\begin{vmatrix} \sigma'_{11} & \sigma'_{12} \\ \sigma'_{21} & \sigma'_{22} \end{vmatrix}$$

Tensor transformation,

New components   Old components

$$\sigma'_{ij} = a_{ik}\, a_{jl}\, \sigma_{kl}$$

Einstein notation twice.

For example σ'_{12} is

$\cos\theta_{11} \cos\theta_{21}$

$$\sigma'_{12} = a_{11}\, a_{21}\, \sigma_{11} + a_{11}\, a_{22}\, \sigma_{12} + a_{11}\, a_{23}\, \sigma_{13}$$
$$+ a_{12}\, a_{21}\, \sigma_{21} + a_{12}\, a_{22}\, \sigma_{22} + a_{12}\, a_{23}\, \sigma_{23}$$
$$+ a_{13}\, a_{21}\, \sigma_{31} + a_{13}\, a_{22}\, \sigma_{32} + a_{13}\, a_{23}\, \sigma_{33}$$

There will be similar expressions for σ'_{11}, σ'_{13}, σ'_{23} .... !

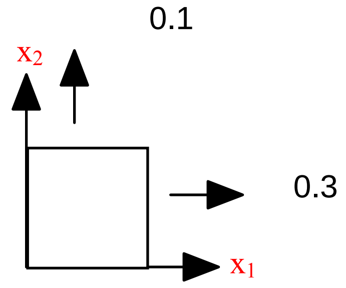
If we now look at another expression, σ'_{13}

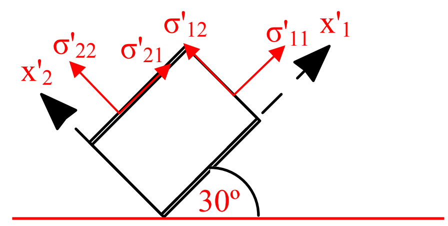
$\sigma'_{13} = a_{11}a_{31}\sigma_{11} + a_{11}a_{32}\sigma_{12} + a_{11}a_{33}\sigma_{13} + a_{12}a_{31}\sigma_{21} + a_{12}a_{32}\sigma_{22} + a_{12}a_{33}\sigma_{23} + a_{13}a_{31}\sigma_{31} + a_{13}a_{32}\sigma_{32} + a_{13}a_{33}\sigma_{33}$

So if you know the direction cosine, and you know the stress components in the old co-ordinate frame, you can use the above equations to calculate the stress components in the new frame.

An example of tensor coordinate transformation.

Lets keep it simple, 2D.

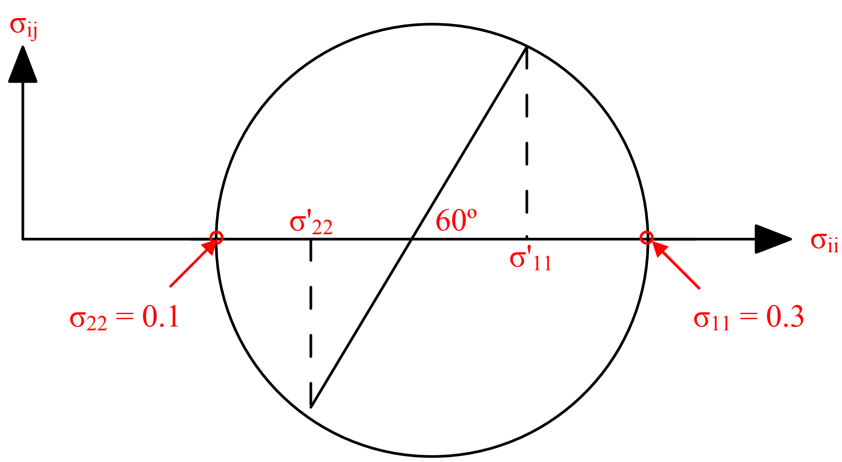


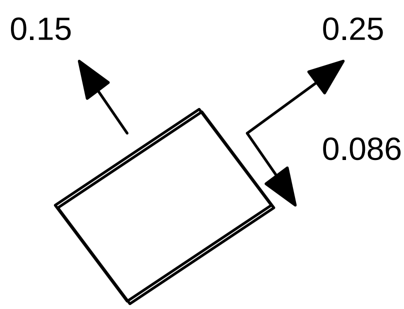


Normal stresses

Stress matrix $\begin{vmatrix} 0.3 & 0 \\ 0 & 0.1 \end{vmatrix}$ $\qquad$ $\begin{vmatrix} \sigma'_{11} & \sigma'_{12} \\ \sigma'_{21} & \sigma'_{22} \end{vmatrix}$

Lets find answer using Mohrs circle.



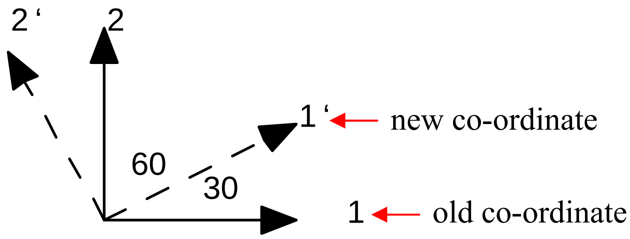


$\begin{vmatrix} 0.25 & -0.086 \\ -0.086 & 0.15 \end{vmatrix}$

Now by tensor transformation.

$$\sigma'_{ij} = a_{ik}\, a_{jl}\, \sigma_{kl}$$ ← Double Einstein notation

Step 1. Direction cosine matrix



as an example
$a_{21} = \cos(90+30)$
$a_{12} = \cos 60$

Old stress matrix

$$\begin{vmatrix} 0.1 & 0 \\ 0 & 0.3 \end{vmatrix}$$

Direction cosine matrix

$$\begin{vmatrix} \cos 30 & \cos 60 & 0 \\ \cos(90+30) & \cos 30 & 0 \\ 0 & 0 & 1 \end{vmatrix} \qquad \begin{vmatrix} 0.866 & 0.5 & 0 \\ -0.5 & 0.866 & 0 \\ 0 & 0 & 1 \end{vmatrix}$$ } Geometry

In our problem the only none zero σ_{kl} are σ_{11} and σ_{22}

So, Remember $\sigma'_{ij} = a_{ik}\, a_{jl}\, \sigma_{kl}$

$$\sigma'_{11} = a_{11}a_{11}\sigma_{11} + a_{12}a_{12}\sigma_{22}$$
$$= (0.866)^2\, 0.3 + (0.5)^2\, 0.1 = 0.245$$

$$\sigma'_{22} = a_{21}a_{21}\sigma_{11} + a_{22}a_{22}\sigma_{22}$$
$$= (-0.5)^2\, 0.3 + (0.866)^2\, 0.1 = 0.15$$

$$\sigma'_{12} = a_{11}a_{21}\sigma_{11} + a_{12}a_{22}\sigma_{22}$$
$$= -(0.866)(0.5)\,0.3 + (0.5)(0.866)0.1 = -0.0866$$

$$\sigma'_{21} = a_{21}a_{11}\sigma_{11} + a_{22}a_{12}\sigma_{22}$$

So stress matrix in new coordinate frame is,

$$\begin{vmatrix} 0.245 & -0.0866 \\ -0.0866 & 0.15 \end{vmatrix}$$ } Stresses in new co-ordinate frame

Mission!

Now we have a feel for tensors we need to develop a mathematical framework that describes stress in the correct way and also the Newtonian flow constitutive equation.

Shear stresses are not a problem!

Normal stresses are a pain because first we must eliminate hydrostatic pressure !

Deviatoric (extra) Stress. The normal stress left after eliminating hydrostatic pressure.

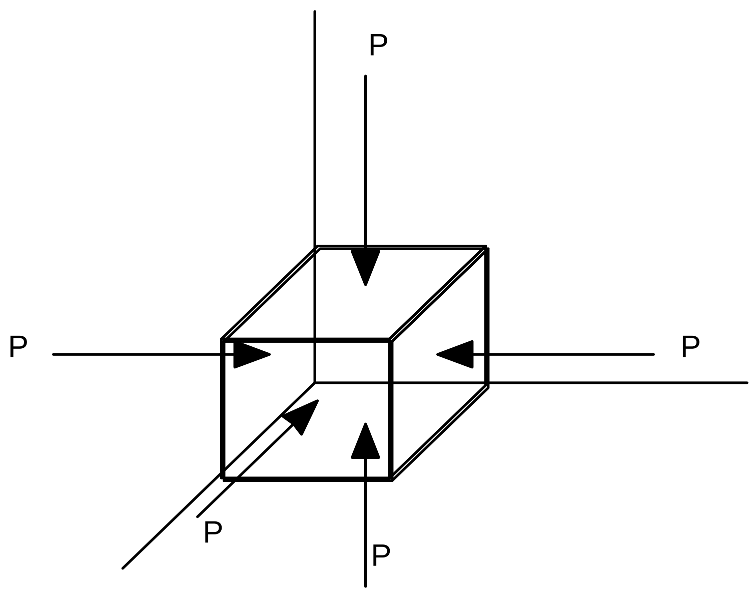
Hydrostatic pressure is isotropic and only acts normal to a face.

Stress matrix.

$$\begin{vmatrix} \mathbf{-p} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{-p} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{-p} \end{vmatrix} = \mathbf{-p.}\ \underline{\mathbf{I}}$$

Stress matrix for pressure

$\underline{\mathbf{I}}$ is identity matrix.
$$\begin{vmatrix} \mathbf{1} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{1} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{1} \end{vmatrix}$$



Define "extra stress" or equivalently "Deviatoric stress"

Total stress = Hydrostatic pressure + Deviatoric stress

Real normal stress (→ Total stress)

Extra stress (→ Deviatoric stress)

$$\underline{\underline{\sigma}} = -p\underline{\underline{I}} + \underline{\underline{\tau}}$$

τ = 0, no flow

$$\begin{vmatrix} \sigma_{11} & \sigma_{12} & - \\ - & - & - \\ - & - & - \end{vmatrix} = \begin{vmatrix} -p & & \\ & -p & \\ & & -p \end{vmatrix} + \begin{vmatrix} \tau_{11} & \tau_{12} & \\ & & \\ & & \end{vmatrix}$$

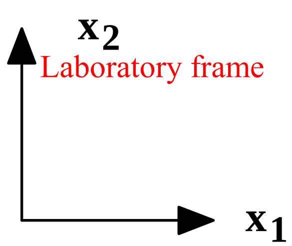
Deviatoric (extra) stress is stress available to deform fluid

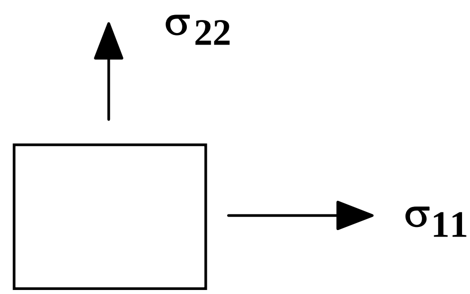
Extra stress → $\boldsymbol{\tau_{ij} = p\, I + \sigma_{ij}}$

τ_{ij} is stress available to deform fluid

Another term we need to understand

Normal Stress Difference (2D deformation)



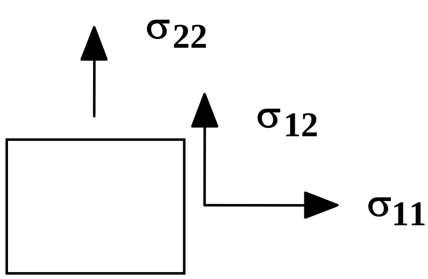


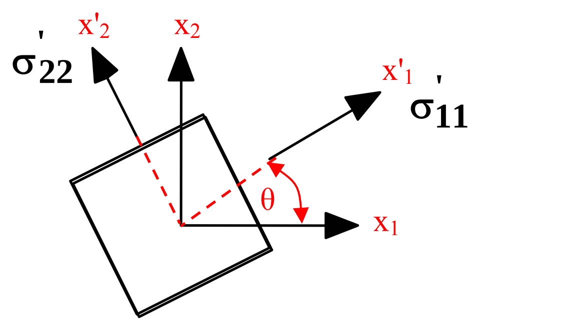
$$\textbf{First Normal Stress Difference} = \sigma_{11} - \sigma_{22}$$

For Newtonian fluid $\sigma_{11} - \sigma_{22} = 0$

Yet another term we need

Principal Stress Difference (2D deformation) $\Longrightarrow$ occurs at a particular co-ordinate frame





$$\begin{vmatrix} \sigma_{11} & \sigma_{12} \\ \sigma_{12} & \sigma_{22} \end{vmatrix} \qquad \begin{vmatrix} \sigma'_{11} & 0 \\ 0 & \sigma'_{22} \end{vmatrix}$$

$\sigma_{ij} \neq 0$

$$\textbf{Principal Stress Difference } \sigma'_{11} - \sigma'_{22}$$

You can use matrix methods, or Mohrs Circle to find magnitude of PSD and orientation of PSD coordinate frame.

Some properties are independent of any choice of coordinates

Invariants of the stress matrix.

Properties that are independent of the coordinate frame chosen.

$$\begin{vmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ \sigma_{12} & \sigma_{22} & \sigma_{23} \\ \sigma_{13} & \sigma_{23} & \sigma_{33} \end{vmatrix}$$

Whatever co-ordinate frame you use

First Invariant I_1

$$I_1 = \text{Trace}\,\underline{\underline{\sigma}} = \sigma_{11} + \sigma_{22} + \sigma_{33}$$

second invariant I_2

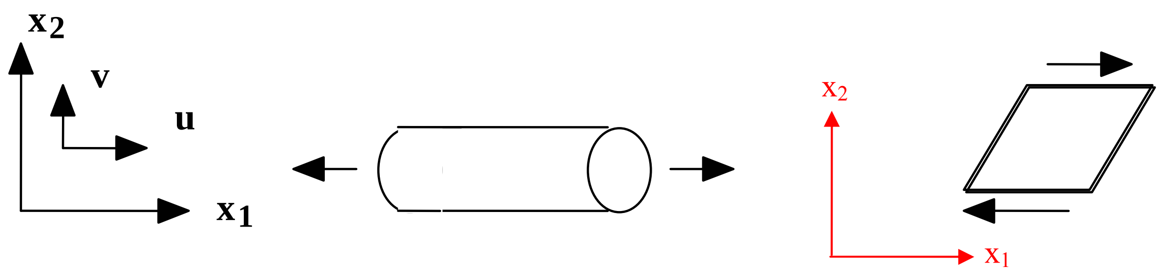
$$I_2 = \sigma_{11}\sigma_{22} + \sigma_{22}\sigma_{33} + \sigma_{11}\sigma_{33} - \sigma_{12}\sigma_{21} - \sigma_{23}\sigma_{32} - \sigma_{13}\sigma_{31}$$

$$\mathrm{I}_2 = \sigma_{11}\sigma_{22} + \sigma_{11}\sigma_{33} + \sigma_{22}\sigma_{33} - \sigma_{12}^2 - \sigma_{23}^2 - \sigma_{13}^2$$

To help us understand generalised deformations, we now have a knowledge of;

1. *A fully generalised description of stress.*
2. *Normal stress differences and Principal stress differences*
3. *Deviatoric or "extra stress"*
4. *Stress invariants*

Types of deformations. **Strain rates.** *(trouble brewing, what about solid body rotation)*



$$\dot{\gamma}_{11} = \frac{d\,u}{d\,x_1} \qquad \dot{\gamma}_{21} = \frac{du}{d\,x_2}$$

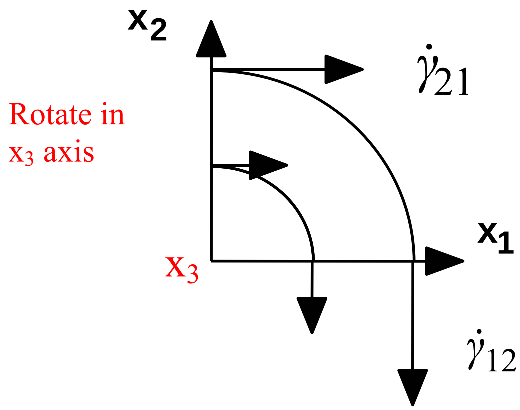
Direction of vel change — Direction of velocity

A strain rate matrix.

$$\begin{matrix} \dot{\gamma}_{11} & \dot{\gamma}_{12} & \dot{\gamma}_{13} \\ \dot{\gamma}_{21} & \dot{\gamma}_{22} & \dot{\gamma}_{23} \\ \dot{\gamma}_{31} & \dot{\gamma}_{32} & \dot{\gamma}_{33} \end{matrix}$$

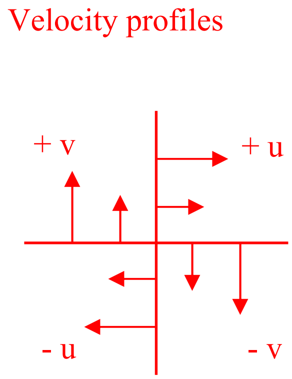
Problem; solid body rotation creates difficulties. Unlike the stress matrix , the strain rate matrix is not symmetric. We have to remove solid body rotation in order to get the deforming strain rate that is of relevance.

Consider solid body rotation on its own.



$$\dot{\gamma}_{21} = \frac{du}{dx_2}$$

$$\dot{\gamma}_{12} : -\frac{dv}{dx_1}$$



For solid body rotation $\dot{\gamma}_{12}$ and $\dot{\gamma}_{21}$ are non zero and so we find a way of "eliminating" solid body rotation from the strain rate matrix.

Contains rotation information

$$\begin{vmatrix} 0 & -\dot{\gamma} & 0 \\ \dot{\gamma} & 0 & 0 \\ 0 & 0 & 0 \end{vmatrix}$$

Non zero strain rate matrix leads to trouble

Separate strain rate matrix into symmetric and anti symmetric components.

Real world strain rate matrix

symmetric component + **anti - symmetric component**

Deformation　　　　Rotation

$$\begin{vmatrix} \dot{\gamma}_{11} & \dot{\gamma}_{12} & \dot{\gamma}_{13} \\ \dot{\gamma}_{21} & \dot{\gamma}_{22} & \dot{\gamma}_{23} \\ \dot{\gamma}_{31} & \dot{\gamma}_{32} & \dot{\gamma}_{33} \end{vmatrix}$$

$$= \begin{vmatrix} \dot{\gamma}_{11} & \frac{1}{2}(\dot{\gamma}_{12}+\dot{\gamma}_{21}) & \frac{1}{2}(\dot{\gamma}_{13}+\dot{\gamma}_{31}) \\ \frac{1}{2}(\dot{\gamma}_{21}+\dot{\gamma}_{12}) & \dot{\gamma}_{22} & \\ & & \dot{\gamma}_{33} \end{vmatrix} + \begin{vmatrix} 0 & \frac{1}{2}(\dot{\gamma}_{12}-\dot{\gamma}_{21}) & \frac{1}{2}(\dot{\gamma}_{13}-\dot{\gamma}_{31}) \\ \frac{1}{2}(\dot{\gamma}_{21}-\dot{\gamma}_{12}) & 0 & \\ & & 0 \end{vmatrix}$$

$$\begin{vmatrix} \dot{\varepsilon}_{11} & \dot{\varepsilon}_{12} & \dot{\varepsilon}_{13} \\ \dot{\varepsilon}_{12} & \dot{\varepsilon}_{22} & \\ & & \dot{\varepsilon}_{33} \end{vmatrix} + \begin{vmatrix} \omega_{11} & \omega_{12} & \omega_{13} \\ \omega_{21} & \omega_{22} & \\ & & \omega_{33} \end{vmatrix}$$

$\dot{\varepsilon}_{ij}$ = symmetric strain rate tensor components.

$\dot{\varepsilon}_{ij} = 0.5(\gamma_{ij} + \gamma_{ji})$

ω_{ij} = anti-symmetric rotation rate tensor compenents.

$\omega_{ij} = 0.5(\gamma_{ij} - \gamma_{ji})$

Examples of symmetric strain rate and rotation rate tensor matrix.

$$\dot{\varepsilon}_{ij} = \frac{1}{2}\left(\dot{\gamma}_{ij} + \dot{\gamma}_{ji}\right) \qquad \omega_{ij} = \frac{1}{2}\left(\dot{\gamma}_{ij} - \dot{\gamma}_{ji}\right)$$

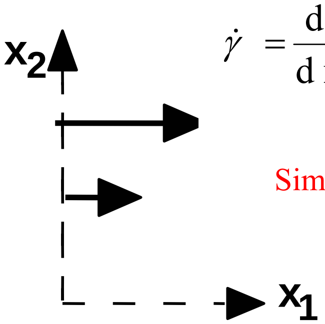
1. Solid body rotation

$$\dot{\gamma}_{\mathrm{ij}} = \begin{vmatrix} 0 & -\dot{\gamma} & 0 \\ \dot{\gamma} & 0 & 0 \\ 0 & 0 & 0 \end{vmatrix} \qquad \dot{\varepsilon}_{\mathrm{ij}} = \begin{vmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{vmatrix} \qquad \omega_{\mathrm{ij}} = \begin{vmatrix} 0 & -\dot{\gamma} & 0 \\ \dot{\gamma} & 0 & 0 \\ 0 & 0 & 0 \end{vmatrix}$$

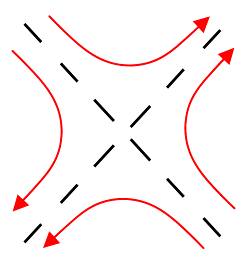
Strain Rate matrix — no deformation — non zero rotation

2. Simple shear $\Rightarrow$ is not so simple !!!!!

$$\dot{\gamma} = \frac{\mathrm{d}\,u}{\mathrm{d}\,x_2}$$



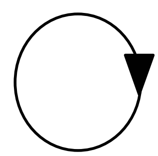
Simple shear =



Pure shear deformation with no rotation

+

Solid body rotation



Strain Rate matrix for simple shear

$$\dot{\gamma}_{\mathrm{ij}} = \begin{vmatrix} 0 & 0 & 0 \\ \dot{\gamma} & 0 & 0 \\ 0 & 0 & 0 \end{vmatrix} \qquad \dot{\varepsilon}_{\mathrm{ij}} = \begin{vmatrix} 0 & \dot{\gamma}/2 & 0 \\ \dot{\gamma}/2 & 0 & 0 \\ 0 & 0 & 0 \end{vmatrix} \qquad \omega_{\mathrm{ij}} = \begin{vmatrix} 0 & -\dot{\gamma}/2 & 0 \\ \dot{\gamma}/2 & 0 & 0 \\ 0 & 0 & 0 \end{vmatrix}$$

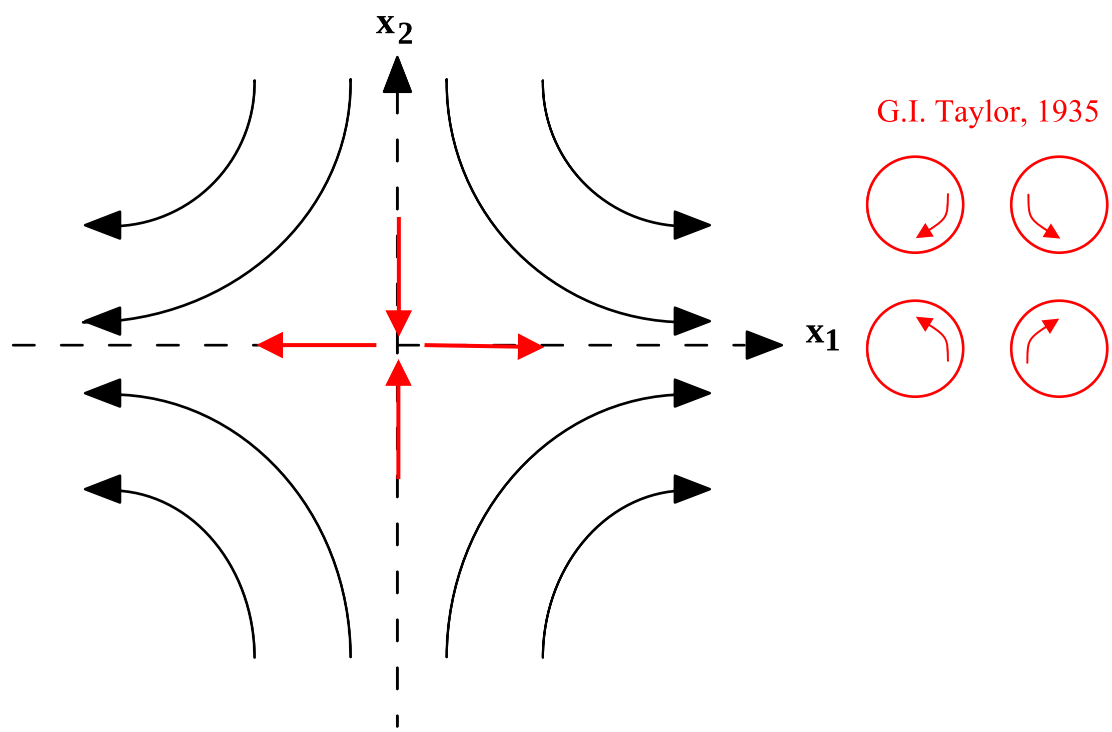
Deformation matrix [non-zero] — Rotation rate matrix [non-zero]

Very important; simple shear is a combination of deformation and rotation.
If you put a cylinder in a simple shear flow it will rotate.

3. Pure shear. A rotation free 2D extensional deformation.

Incompressibility.

$$\mathbf{\mathit{div}\ \underline{u} = \frac{du}{dx_1} + \frac{dv}{dx_2} + \frac{dw}{dx_3} = \dot{\gamma}_{11} + \dot{\gamma}_{22} + \dot{\gamma}_{33} = 0}$$



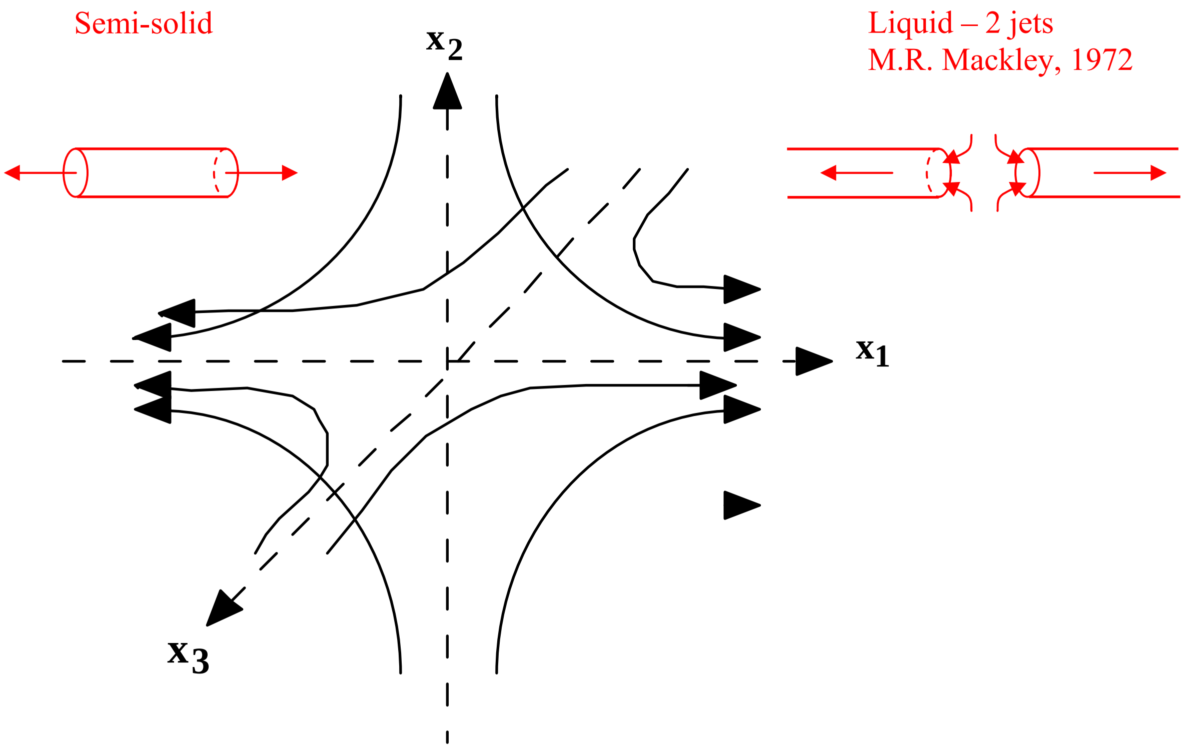
$$\dot{\gamma}_{ij} = \begin{vmatrix} \dot{\gamma} & 0 & 0 \\ 0 & -\dot{\gamma} & 0 \\ 0 & 0 & 0 \end{vmatrix} \qquad \dot{\varepsilon}_{ij} = \begin{vmatrix} \dot{\gamma} & 0 & 0 \\ 0 & -\dot{\gamma} & 0 \\ 0 & 0 & 0 \end{vmatrix} \qquad \omega_{ij} = \begin{vmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{vmatrix}$$

None zero deformation

no rotation

This deformation is elegant as it contains no rotational components.

4. Uniaxial Extension. A rotation free 3D extensional deformation.



If the fluid is incompressible, div u = $\gamma_{11} + \gamma_{22} + \gamma_{33} = 0$

By symmetry, $\gamma_{22} = \gamma_{33}$, so $\gamma_{22} = \gamma_{33} = -0.5\gamma_{11}$

$$\dot{\gamma}_{11} + \dot{\gamma}_{22} + \dot{\gamma}_{33} = 0$$

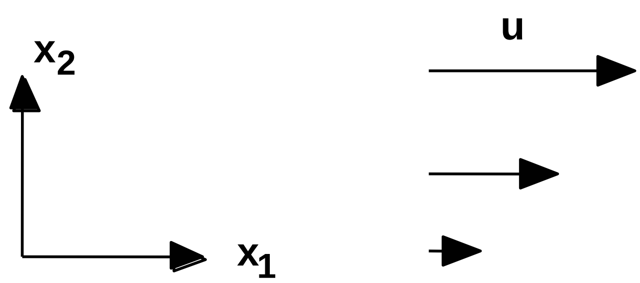
$$\dot{\gamma}_{ij} = \begin{vmatrix} \dot{\gamma} & 0 & 0 \\ 0 & -\dot{\gamma}/2 & 0 \\ 0 & 0 & -\dot{\gamma}/2 \end{vmatrix} \qquad \dot{\varepsilon}_{ij} = \begin{vmatrix} \dot{\gamma} & 0 & 0 \\ 0 & -\dot{\gamma}/2 & 0 \\ 0 & 0 & -\dot{\gamma}/2 \end{vmatrix} \qquad \omega_{ij} = \begin{vmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{vmatrix}$$

Most engineering flows are combinations of simple shearing and rotation free extensional flows

Now we are going to define a Newtonian fluid properly!

Generalised definition of Newtonian flow.

Remember an old friend. Simple shearing flow.



$$\dot{\gamma}_{21} = \frac{du}{dy}$$

$$\begin{vmatrix} 0 & 0 & 0 \\ \dot{\gamma} & 0 & 0 \\ 0 & 0 & 0 \end{vmatrix}$$

$$\tau = \eta\, \dot{\gamma}_{21}$$

We need our generalised definition to be consistent with the above.
Generalised definition.

Stress available to deform liquid, (deviatoric extra stress) $\boldsymbol{\tau_{ij}}$ proportional to rotation free strain rate $\dot{\varepsilon}_{ij}$

$$\tau_{ij} = 2\eta\, \dot{\varepsilon}_{ij}$$

Real stress → $\underline{\underline{\sigma}} + \underline{\underline{I}}\, P = 2\,\eta\, \underline{\underline{\dot{\varepsilon}}}$ ← pressure

factor of 2 ensures self consistency with $\tau = \eta\dot{\gamma}$

extension

$$\textit{if } i = j \text{ then } \quad \tau_{ii} = \sigma_{ii} + P = 2\eta\, \dot{\varepsilon}_{ii}$$

$$\textit{if } i = j = 1, \text{ then } \quad \tau_{11} = \sigma_{11} + P = 2\eta\, \dot{\varepsilon}_{11}$$

shear

$$\textit{if } i \neq j \text{ then } \quad \tau_{ij} = \sigma_{ij} = 2\eta\, \frac{1}{2}\left(\dot{\gamma}_{ij} + \dot{\gamma}_{ji}\right)$$

For simple shearing flow.

$$\tau_{21} = \sigma_{21} = 2\eta \frac{1}{2}\left(0 + \dot{\gamma}_{21}\right) = \eta\,\dot{\gamma}_{21}$$ → Self consistent with "normal defn of viscosity

$\tau = \eta\,\dot{\gamma}$ as before, so our generalised definition is self consistent with our simple definition

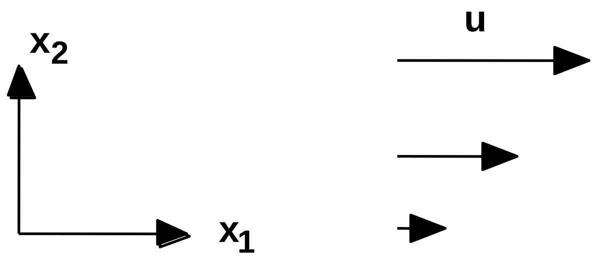
So generalised definition of a Newtonian flow is

$$\underline{\underline{\sigma}} + \underline{\underline{I}}\,P = 2\,\eta\,\underline{\underline{\dot{\varepsilon}}}$$

where

(Lets understand simple shear)

Simple shear Newtonian flow.

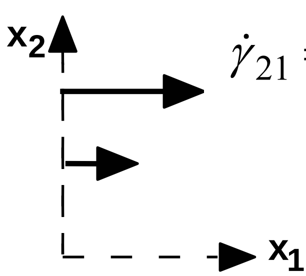


$$\dot{\gamma}_{21} = \frac{d\,u}{dx_2}$$

$$\sigma_{21} = \eta\,\dot{\gamma}_{21}$$

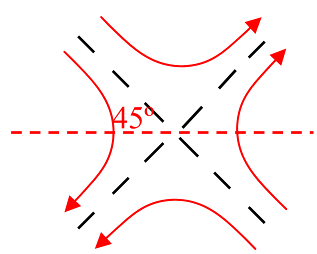
$$\dot{\gamma}_{ij} = \begin{vmatrix} 0 & 0 & 0 \\ \dot{\gamma} & 0 & 0 \\ 0 & 0 & 0 \end{vmatrix} \qquad \dot{\varepsilon}_{ij} = \begin{vmatrix} 0 & \dot{\gamma}/2 & 0 \\ \dot{\gamma}/2 & 0 & 0 \\ 0 & 0 & 0 \end{vmatrix} \qquad \omega_{ij} = \begin{vmatrix} 0 & -\dot{\gamma}/2 & 0 \\ \dot{\gamma}/2 & 0 & 0 \\ 0 & 0 & 0 \end{vmatrix}$$

Strain rate — rotation free — rotation



$$\dot{\gamma}_{21} = \frac{d\,u}{dx_2}$$

=



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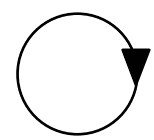
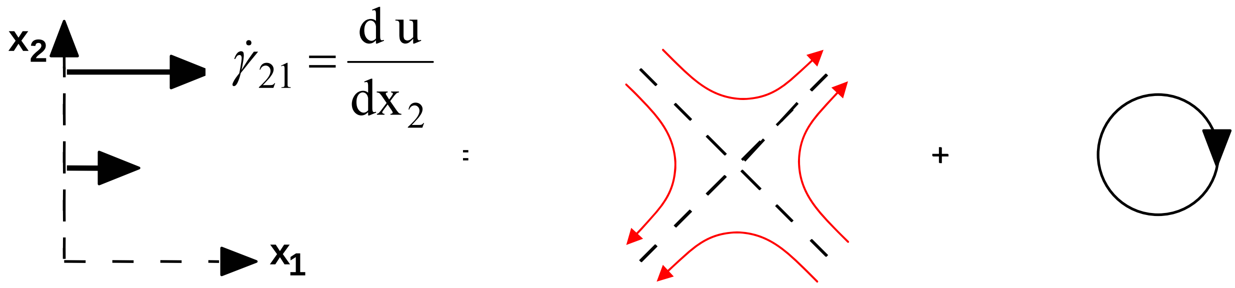
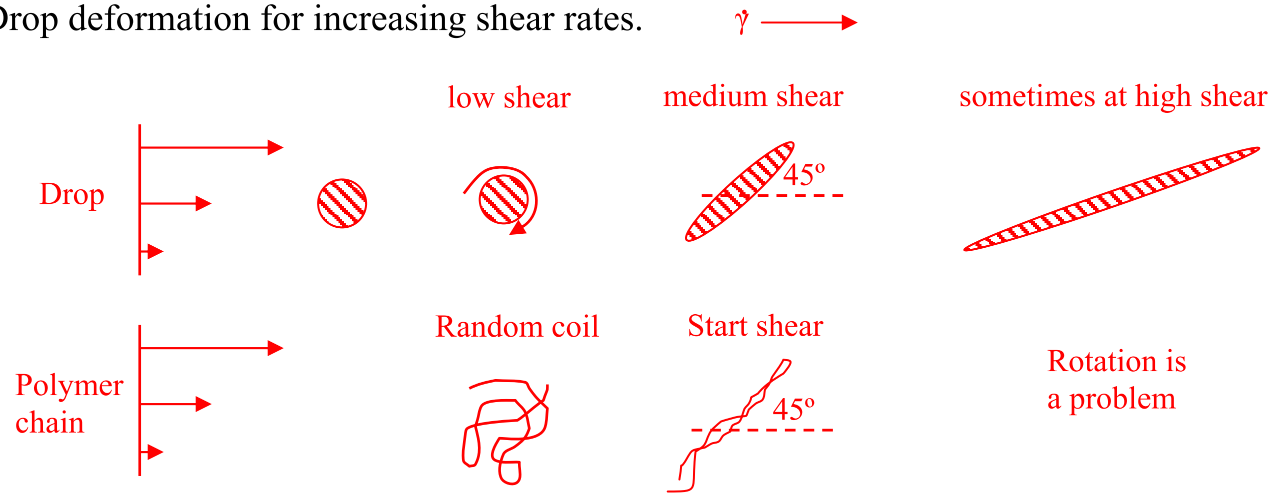


Simple shear is a combination of deformation and rotation. This has concequences in relation to way liquid drops and polymer chains deform, stretch and in some cases break during flow.

Droplet (and polymer chain) deformation in simple shearing flow.



Drop deformation for increasing shear rates.



The "Grace diagram".

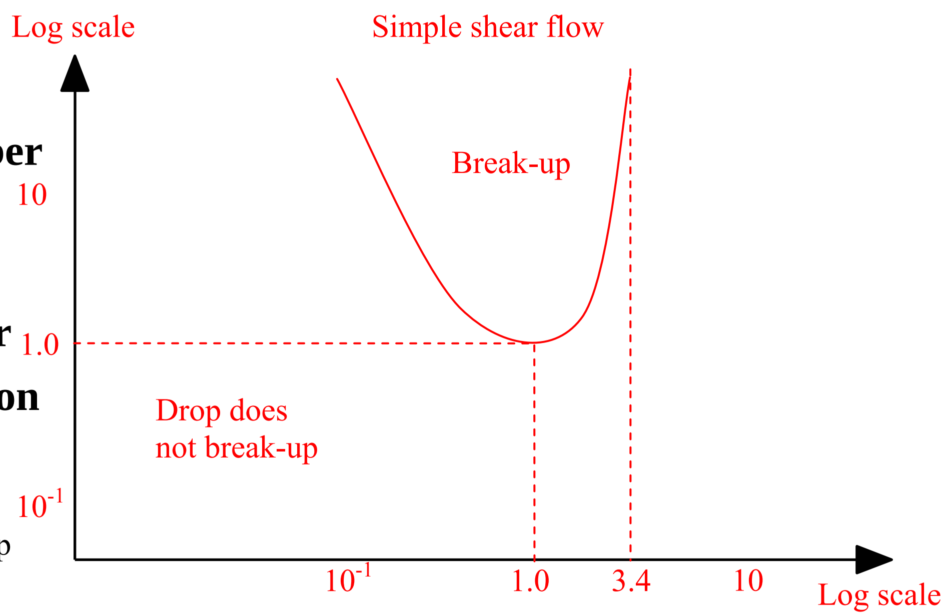
Capillary Number

$$Ca = \frac{\dot{\gamma}\,\eta\, D}{\nu}$$

D = drop diameter

υ = surface tension

Capillary number is the ratio of the viscous stresses acting to break up the droplet and to the surface tension acting to hold the droplet together

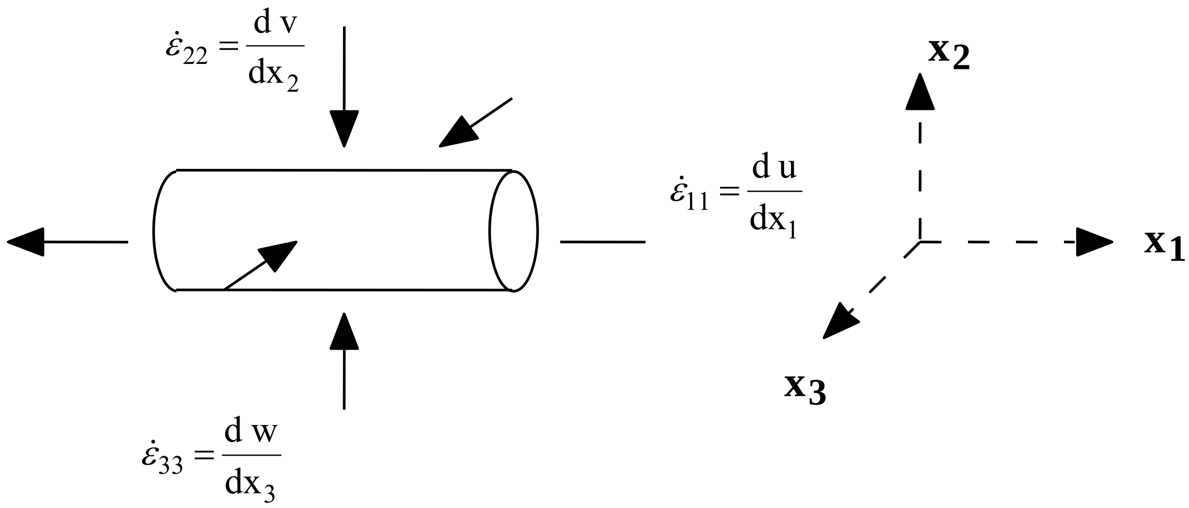


$$\textbf{viscosity ratio} = \frac{\eta_{\text{dispersed drop}}}{\eta_{\text{continuous phase}}}$$

The effect of rotation on the rheology and microstructure of certain fluids.

(Lets look at "extensional flows")

Uniaxial Newtonian extensional flow.



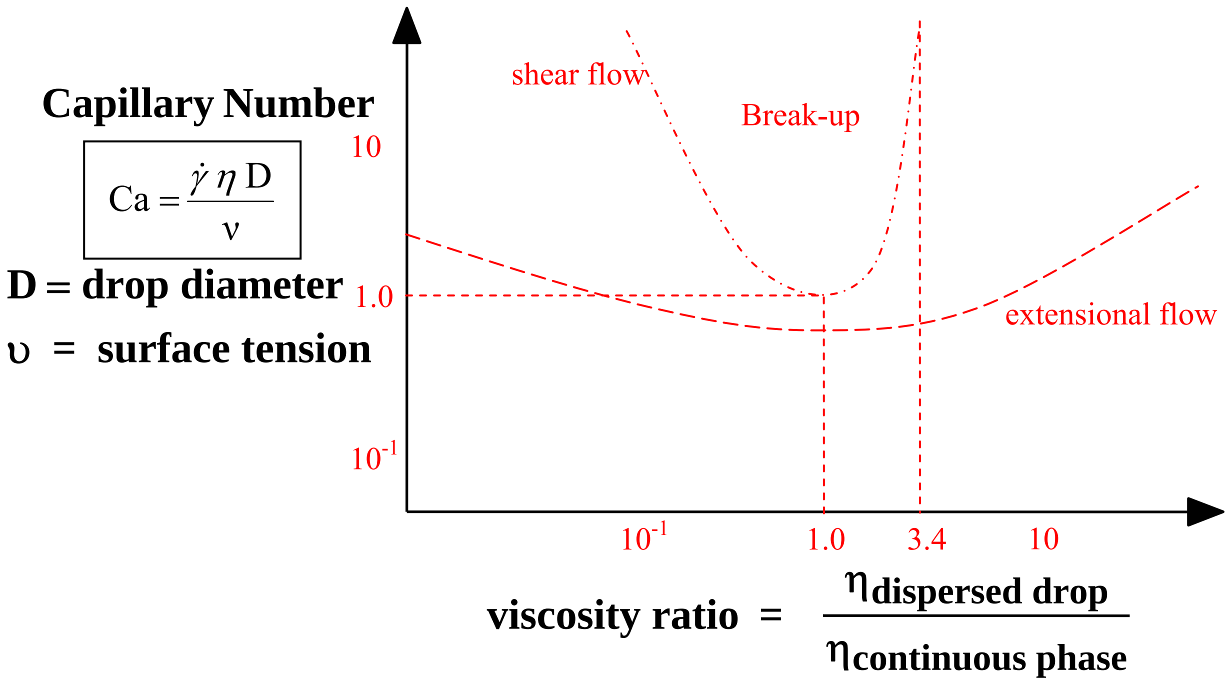
strain rate matrix

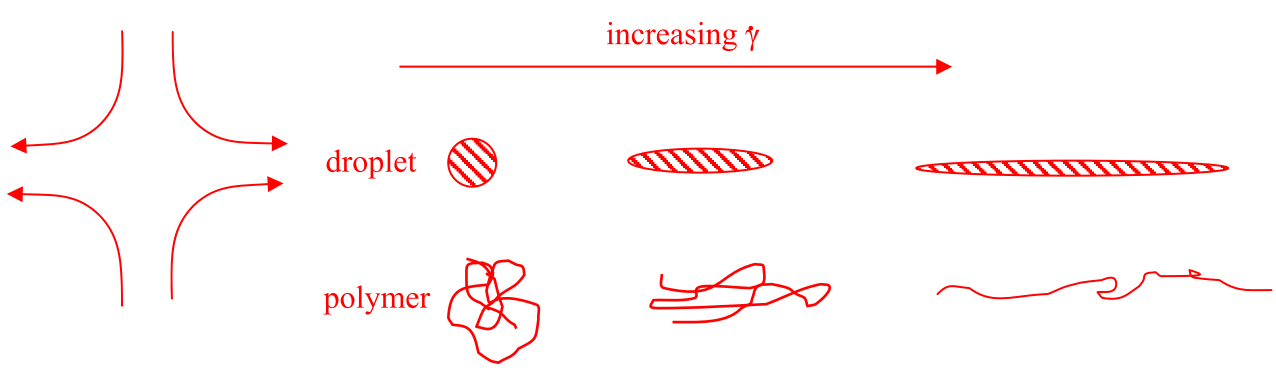
$$\begin{vmatrix} \dot{\varepsilon} & 0 & 0 \\ 0 & -\frac{\dot{\varepsilon}}{2} & 0 \\ 0 & 0 & -\frac{\dot{\varepsilon}}{2} \end{vmatrix}$$

No rotation

Pure extensional flows contain no rotation and they are more powerful at stretching drops and stretching polymer chains.

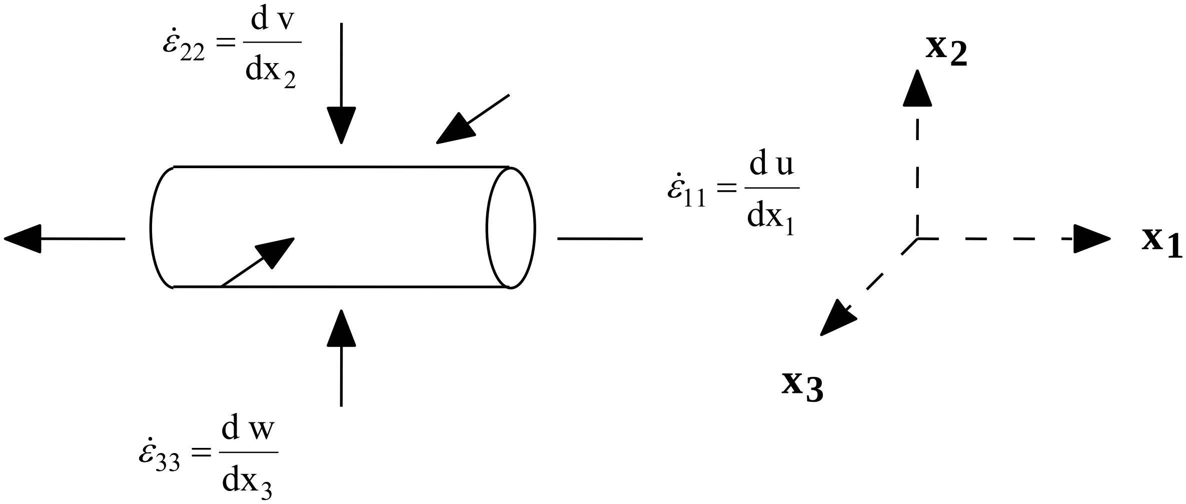
Droplet deformation in extensional flows. The “Grace diagram”.





Simple shear :- poor for deformation
Extensional flow :- efficient for deformation

Uniaxial Newtonian extensional flow.   **The Trouton viscosity ratio**



strain rate matrix

$$\begin{vmatrix} \dot{\varepsilon} & 0 & 0 \\ 0 & -\frac{\dot{\varepsilon}}{2} & 0 \\ 0 & 0 & -\frac{\dot{\varepsilon}}{2} \end{vmatrix}$$

$$\sigma_{ii} + \mathrm{P} = 2\eta\,\dot{\varepsilon}_{\mathrm{ii}}$$

$$\sigma_{11} + \mathrm{P} = 2\eta\,\dot{\varepsilon}$$

$$\sigma_{22} + \mathrm{P} = -2\eta\,\frac{\dot{\varepsilon}}{2}$$

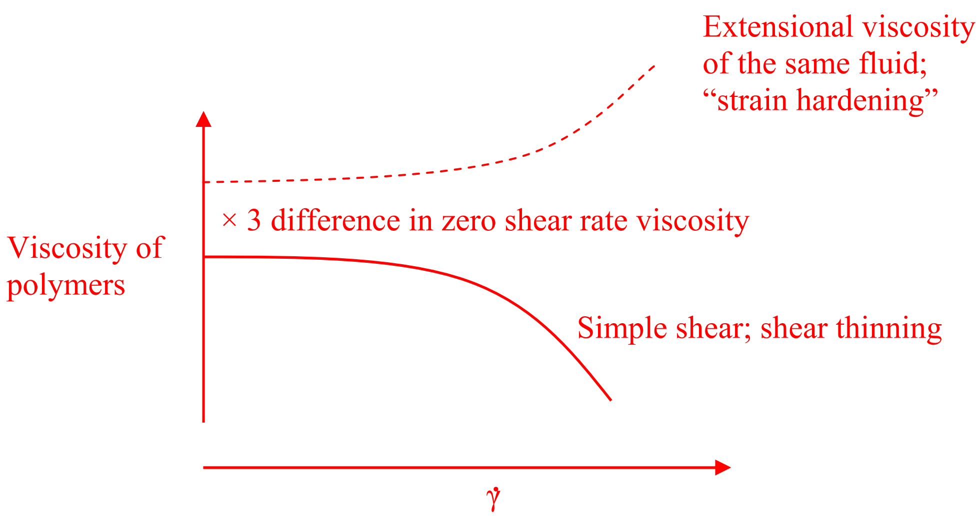
$$\sigma_{11} - \sigma_{22} = 3\eta\,\dot{\varepsilon} = \eta_{\mathrm{e}}\,\dot{\varepsilon}$$

The Trouton Ratio $\dfrac{\eta_e}{\eta} = 3$

The effect of chain stretching on simple shear and extensional viscosities.

Simple shear; chains stretch in flow; reduces viscosity = shear thinning

Extensional flow; chains stretch in flow; increases extensional viscosity= "strain hardening". This can have important consequences for certain processes. Polymer Extensional viscosity



Finally.

The generalised form of the Wagner Integral Viscoelastic equation.

Before.

$$\tau(t) = -\int_{-\infty}^{t} \sum_{i} \frac{g_i}{\lambda_i}\, e^{-\left(t-t'\right)/\lambda_i}\, e^{-k\left|\gamma\left(t,t'\right)\right|}\, \gamma\left(t,t'\right) dt'$$

→ scalar form

Now, generalised stress components given by,

$$\tau_{ij}(t) = -\int_{-\infty}^{t} \sum_{i} \frac{g_i}{\lambda_i}\, e^{-\left(t-t'\right)/\lambda_i}\, h\left(I_1, I_2\right) C^{-1}\left(t,t'\right) dt'$$

$\tau_{ij}(t)$: 6 components

$h(I_1, I_2)$: damping factor – invariant to co-ordinate rotation

$C^{-1}(t,t')$: Tensor strain deformation

$C^{-1}(t,t')$ is the "Finger" strain tensor

(See Dealy and Wissbrum if interested, but you don't need to know details of this strain tensor)

$h(I_1, I_2)$ is the damping factor

where I_1 and I_2 are the first and second invariants of the strain Finger tensor.

$$h(I_1, I_2) = e^{-k\left(\beta I_1 + (1-\beta) I_2 - 3\right)}$$

So now we now have our generalised Visco-elastic constitutive equation that can be solved numerically for engineering problems using for example Fluent "Polyflow". This means that complex engineering problems can now be modelled with realistic viscoelastic constitutive equations