Section 2.

The application of Non Newtonian constitutive equations to simple engineering flow geometries.

Application of simple constitutive equations to pipe /capillary and channel flow.

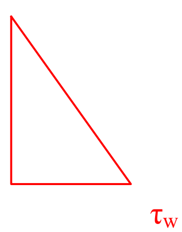
Pipes / capillaries. Easy experimentally, however results can be difficult to interpret. Problem is close to many engineering situations.

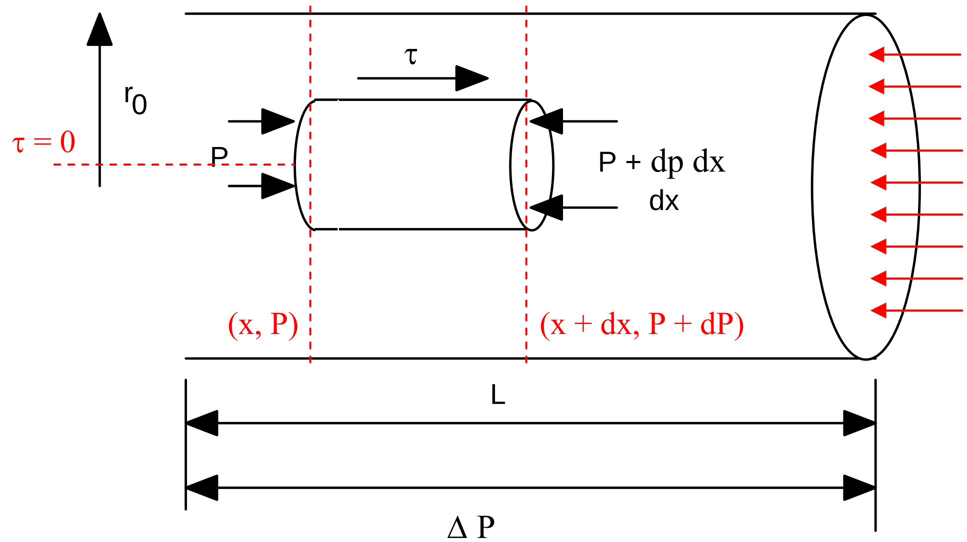
Relevance. 1) Engineering calculations, 2) capillary rheometry, 3) process understanding.

Laminar Newtonian flow $Re < \approx 2,000$

(Revision)

$Re = \rho\, u\, 2\, r / \eta$





Force balance

$$\tau\, 2\pi\, r dx \;-\; \frac{dP}{dx} dx\, \pi\, r^2 \;=\; 0$$

General Result $$\tau \;=\; \frac{r}{2}\,\frac{dP}{dx} \;=\; \frac{r}{2}\,\frac{\Delta P}{L}$$ $\tau = \frac{\Delta P\, r}{2L}$ Shear stress is a linear function of radius

At wall $$\tau_o \;=\; \frac{r_o}{2}\,\frac{\Delta P}{L}$$ } *No difficulty here*

The link between pressure drop and wall shear stress

Newtonian Constitutive equation

$$\tau = \eta\dot{\gamma} = \eta\frac{du}{dr} = \frac{r}{2}\frac{\Delta P}{L}$$

Integrate with u = 0 at r = r_o

(Assume No slip at wall – care sometimes needed here. We could add a slip velocity. See past Tripos question)

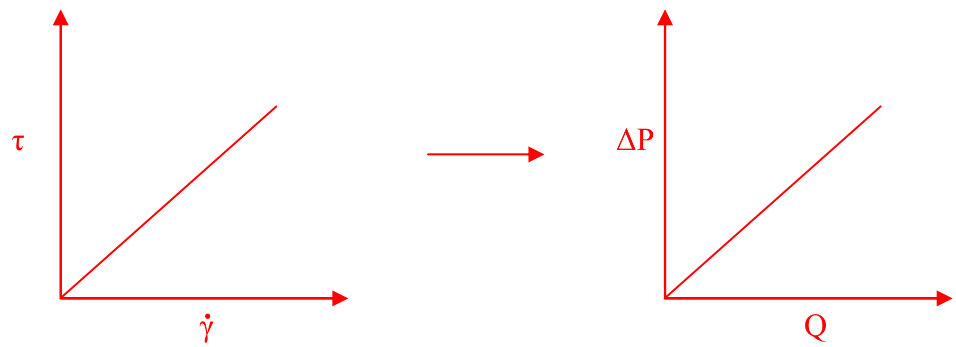
Some high viscosity fluids can slip at the wall, they take the path of least resistance.

Yields $$u(r) = -\frac{\Delta P}{4L\eta}\left[r_o^2 - r^2\right]$$ Parabolic profile

Volumetric flow $$Q = 2\pi\int_o^{r_o} r\,u(r)\,dr$$

Yields $$\eta = \frac{\pi\, r_o^4\, \Delta P}{8\,L\,Q}$$

$$\Delta P = \frac{8\eta L}{\pi r_o^4} Q$$



$\Delta P \;\alpha\; \eta$
$\Delta P \;\alpha\; L$
$\Delta P \;\alpha\; r_o^{-4}$

As seen, the pressure drop is very sensitive to r. Hence in rheology it is crucial to know your geometry with precision

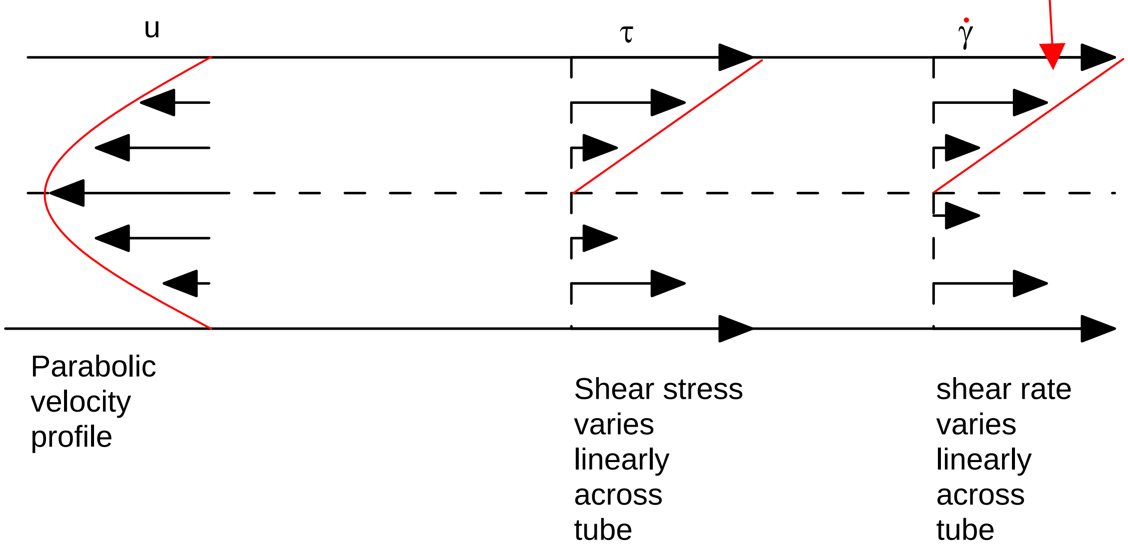
Laminar, Newtonian and Linear

Note……

Shear rate $\dot{\gamma} = \frac{du}{dr} = \frac{\Delta P}{2L\eta} r = \frac{4\,Q}{\pi\, r_o^4} r$

shear rate, linear with r for a Newtonian flow

at wall, $\dot{\gamma}_o = \frac{\mathbf{4\,Q}}{\pi\,\mathbf{r_o^3}}$ (for Newtonian fluid)

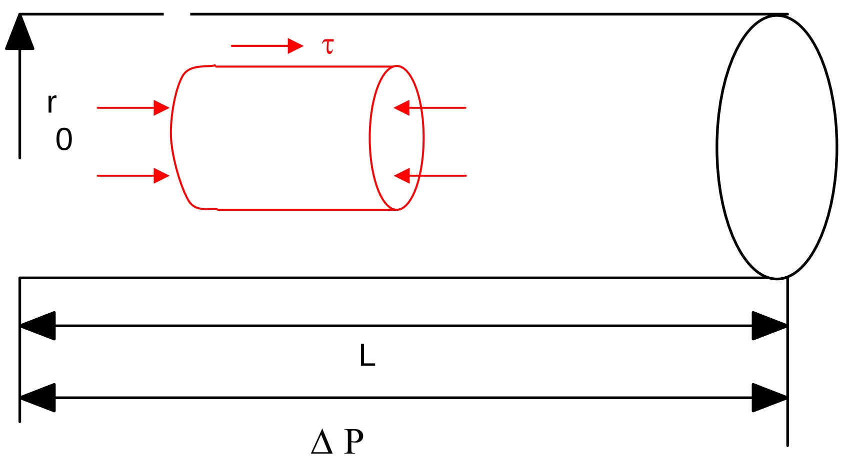


τ and $\dot{\gamma}$ vary across capillary, this can lead to difficulties in interpretation for Non Newtonian fluids where $\eta\left(\dot{\gamma}\right)$.

Capillary is a "variable stress Rheometer", results in non-uniform shear rate rheometer with complication for Non-Newtonian Fluids

Capillary/Pipe flow is potentially "complex" rheologically, because the strain rate/stress is not constant across the capillary/pipe.

Laminar pipe / Capillary flow of power law fluid



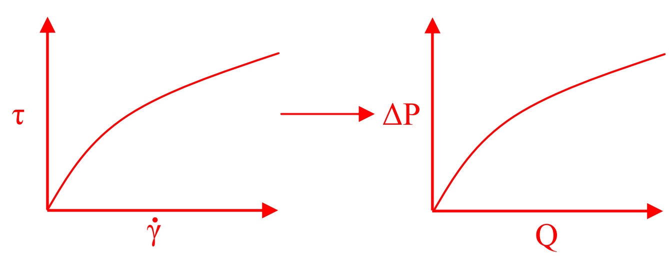
Force balance, (as before) $\tau = \frac{r}{2}\frac{\Delta P}{L}$

Constitutive equation, $\tau = k\,\dot{\gamma}^n = k\left[\frac{du}{dr}\right]^n = \frac{r}{2}\frac{\Delta P}{L}$

Integrate assuming, u = o at $r = r_o$

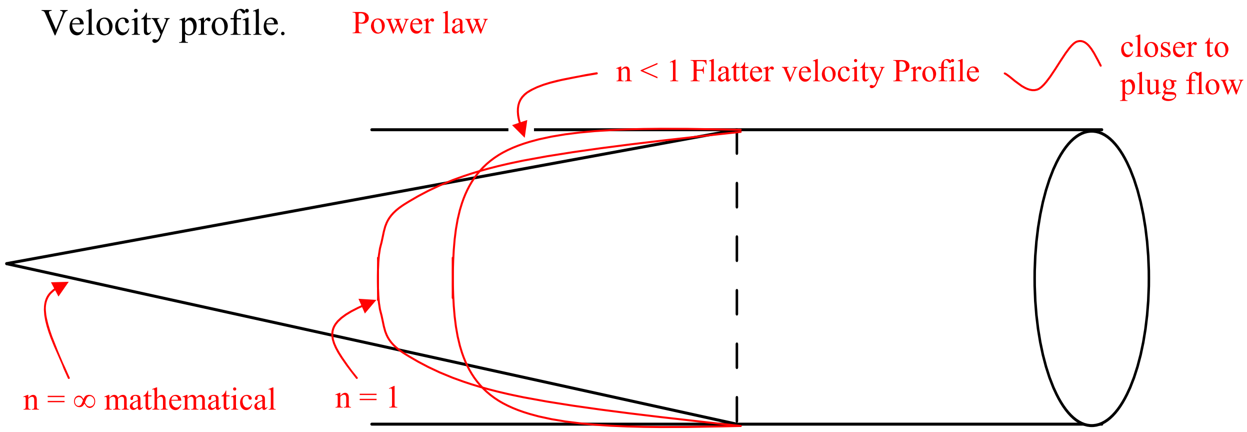
$$u(r) = -\left[\frac{\Delta P}{2L\,k}\right]^{1/n}\frac{n}{n+1}\left[r_0^{\frac{n+1}{n}} - r^{\frac{n+1}{n}}\right] \quad m/s$$

volumetric flow Q

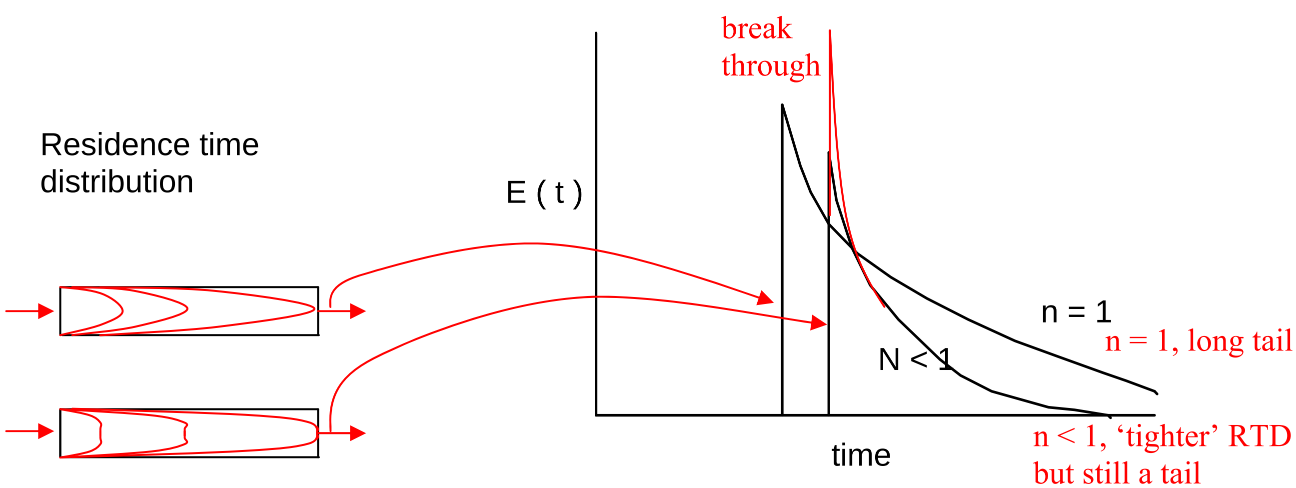


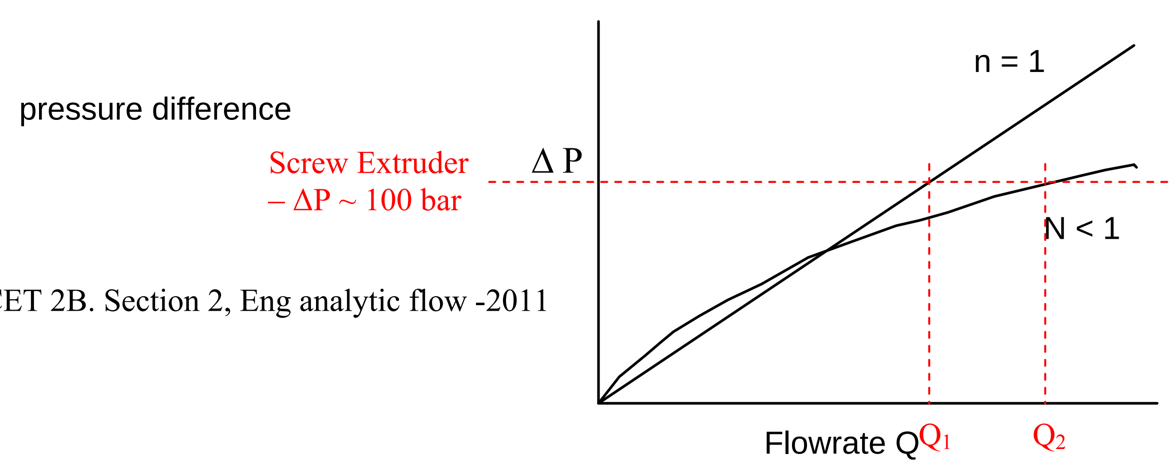
$$\Delta P = 2\,L\,k\left[\frac{(3n+1)}{\pi\, n\, r_0^{\frac{3n+1}{n}}}\right]^n Q^n$$

Velocity profile.



n < 1 Shear thinning, Velocity profile is flatter, generally goods news. Sharper RTD, but still low velocity components at wall. HT and MT correlations not significantly different to Newtonian



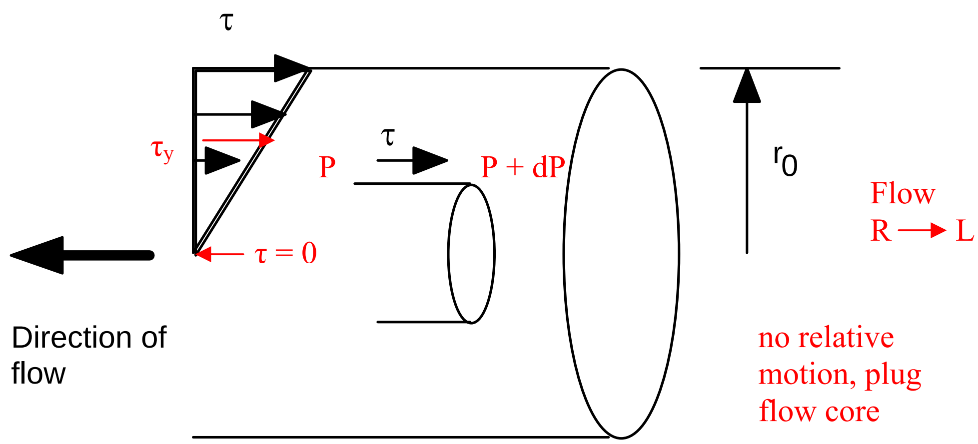


Laminar pipe flow of Bingham plastic

$\tau = \pm\, \tau_y + \eta\dot{\gamma}$

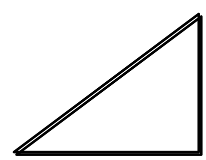
Force balance

$\tau = r\,\Delta P/2L$



$\tau > \tau_y \qquad \tau = \pm\, \tau_y + \eta\dot{\gamma}$

$\tau < \tau_y \qquad \dot{\gamma} = 0$



pressure gradient

Constitutive equation

In region where there is relative fluid motion

$$\tau = \pm\, \tau_y + \eta\dot{\gamma}$$

scalar, need to assign the correct sign

check τ_y acts against motion of fluid

$$\tau = \tau_y + \eta\dot{\gamma} = \tau_y + \eta\frac{du}{dr} = \frac{r}{2}\frac{\Delta P}{L}$$

assume when $r = r_o$, $u = 0$, no slip bc

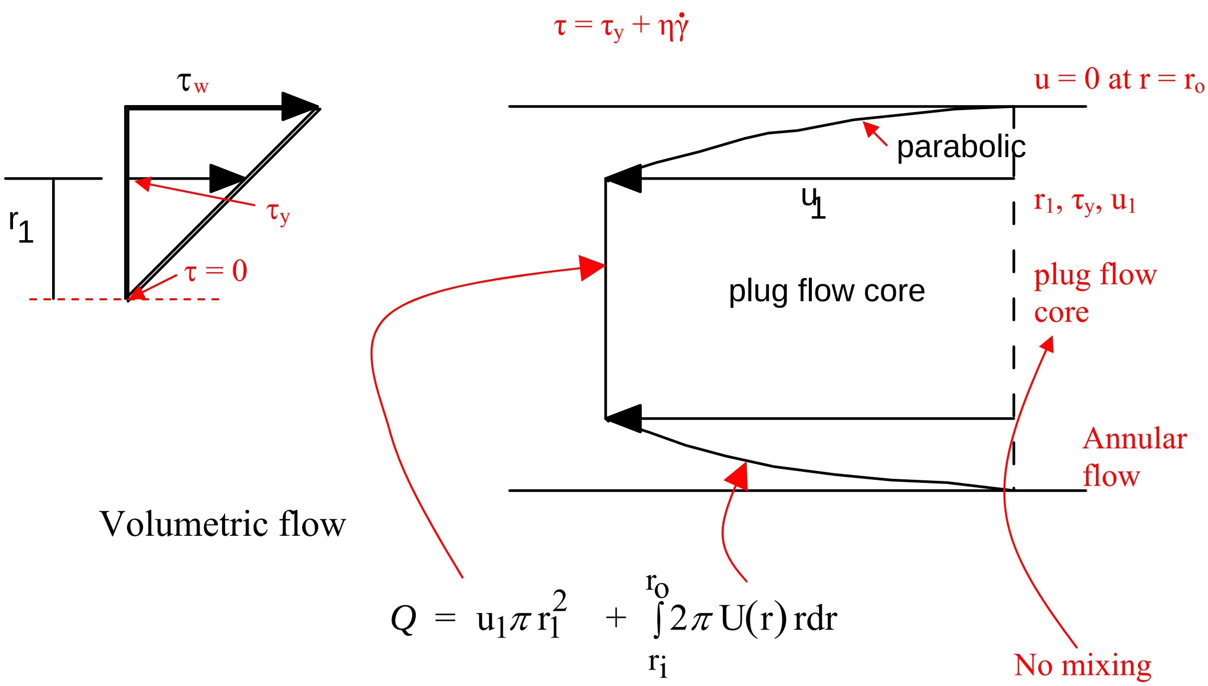
$$u = -\frac{\Delta P}{4L\eta}\left[r_o^2 - r^2\right] + \frac{\tau_y}{\eta}\left[r_o - r\right] \qquad \text{for } \tau > \tau_y$$

(check eqn. let $\tau_y = 0$, then Newtonian profile, OK)

when $\tau < \tau_y$ no relative shear i.e., $\dot{\gamma} = 0$

$$\therefore \frac{du}{dr} = 0 \qquad \text{i.e., Plug flow}$$

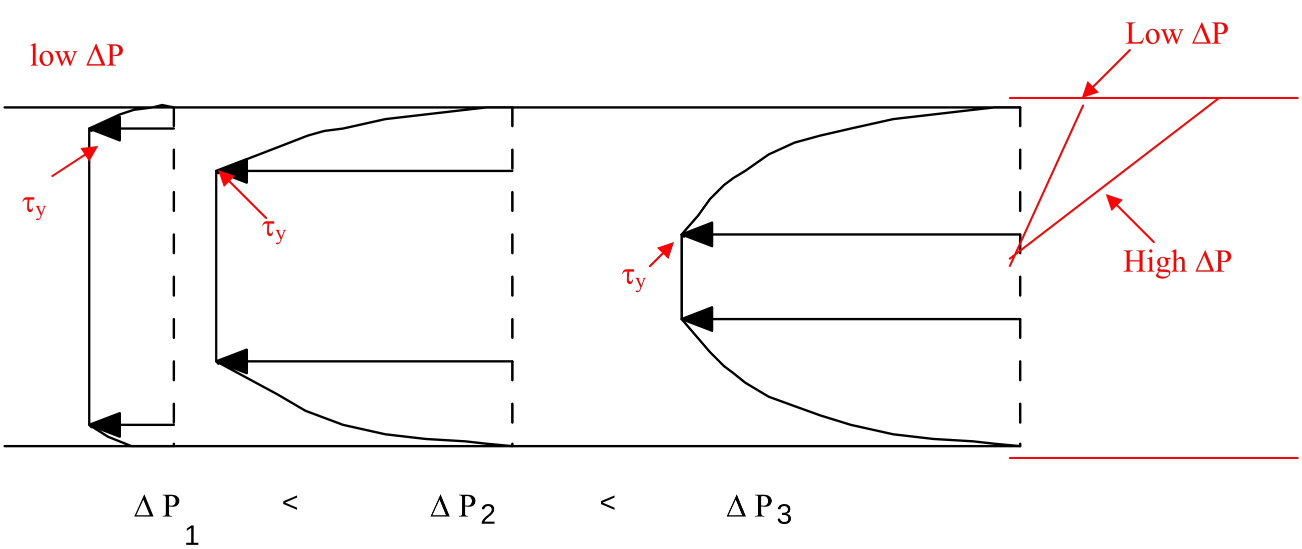
$$\tau = \frac{r}{2}\frac{\Delta P}{L} \qquad \therefore r_1 = 2\tau_y\frac{L}{\Delta P}$$



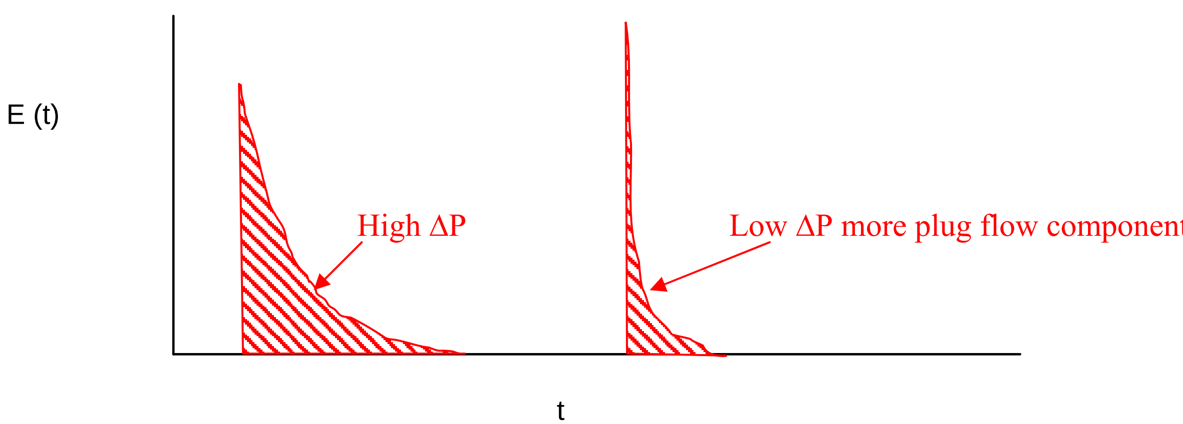
Volumetric flow

$$Q = u_1 \pi r_1^2 + \int_{r_i}^{r_o} 2\pi U(r)\, r\, dr$$

The effect of changing ΔP, for given τ_y (ex toothpaste)



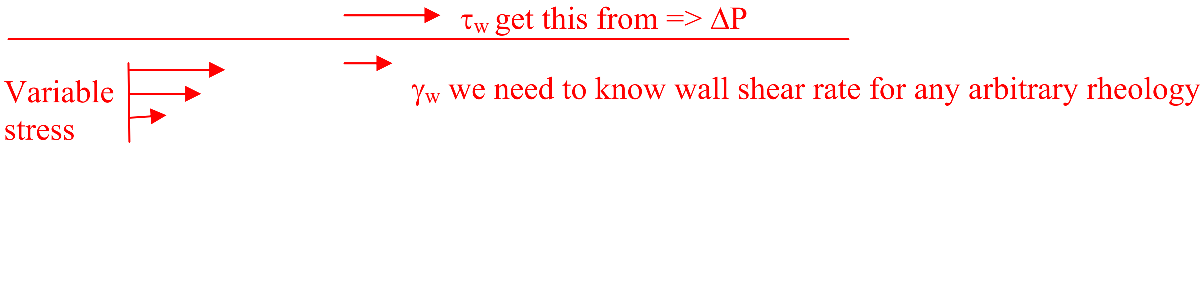
conclusion. Flow is more plug flow at low ΔP



Pipe flow complications. (we consider two). Rheology

Capillaries sometimes are used as a rheometer . Issues to consider

1. What is the shear rate at the wall?



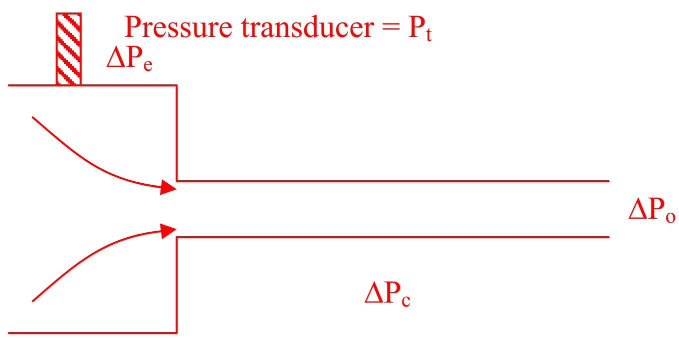
$$\tau = \eta_a \dot{\gamma}$$

Measure τ => τ_w

$\dot{\gamma}_w$ => we need to know this

Obtain apparent viscosoty

2. Entry flow pressure drop.



$$\Delta P_t = P_t - P_o = \Delta P_e + \Delta P_c$$

Worry about entry pressure drop ΔP_e

Use pipe flow ΔP (Q) to identify form of constitutive equation.

Measure pressure drop (ΔP) as function of volumetric flowrate (Q)

Rabinowitsch Correction

We need to know shear stress and shear rate at some location. $\tau = \eta_a \dot{\gamma}$

Go to the wall!

$$\tau_o = \frac{r_o}{2}\frac{\Delta P}{L} \ldots (1)$$ Fine!

We now need $\dot{\gamma}_o$ .shear rate at wall More difficult, but we know Q

$$Q = 2\pi \int_o^{r_o} r\, u(r)\, dr = 2\pi \left\{ \left[\frac{r^2}{2} u(r) \right]_o^{r_o} - \int_o^{r_o} \frac{r^2}{2} du \right.$$

integrate by parts assumes no slip at the wall

$$BC \quad r = r_o, \quad u = 0$$

$$Q = -\pi \int_o^{r_o} r^2 \frac{du}{dr} dr$$

Change variables! Express $\int$ in terms τ

Now $$\frac{r}{r_o} = \frac{\tau}{\tau_o}$$ $\tau = \frac{\tau_o r}{r_o}$

So $$\boldsymbol{Q} = -\pi \frac{r_o^3}{\tau_o^3} \int_o^{\tau_o} \dot{\gamma}\, \tau^2 d\tau$$ ← expressed in terms of τ

$$\frac{1}{\pi r_o^3} Q\, \tau_o^3 = -\int_o^{\tau_o} \dot{\gamma}\, \tau^2 d\tau$$

Differentiate , wrt τ_o, note Q is a function of τ_o

$$\frac{1}{\pi r_o^3}\left[\tau_o^3 \frac{dQ}{d\tau_o} + 3\tau_o^2 Q \right] = -\dot{\gamma}_o \tau_o^2$$

note $$\tau_o = \frac{r_o}{2}\frac{\Delta P}{L}$$

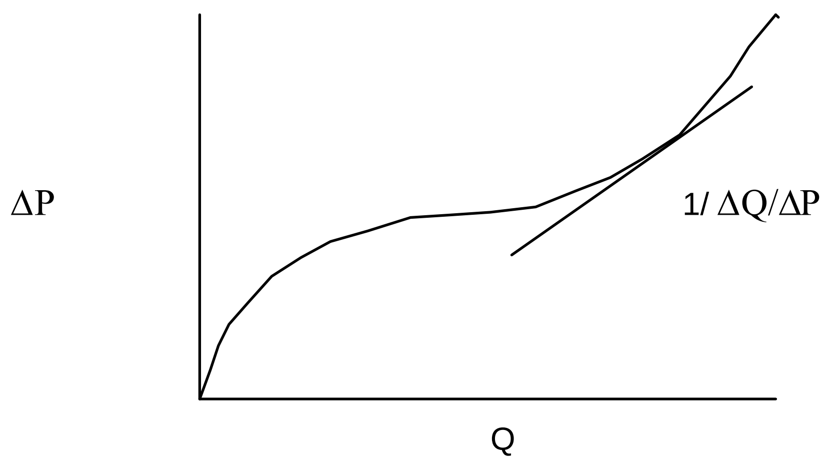
shear rate at the wall

$$-\dot{\gamma}_o = \frac{1}{\pi r_o^3}\left[3Q + \Delta P \frac{dQ}{d(\Delta P)}\right] \ldots. (2)$$

volumetric flow rate (Q)
differential of Q Vs ΔP

inverse slope of ΔP(Q) curve

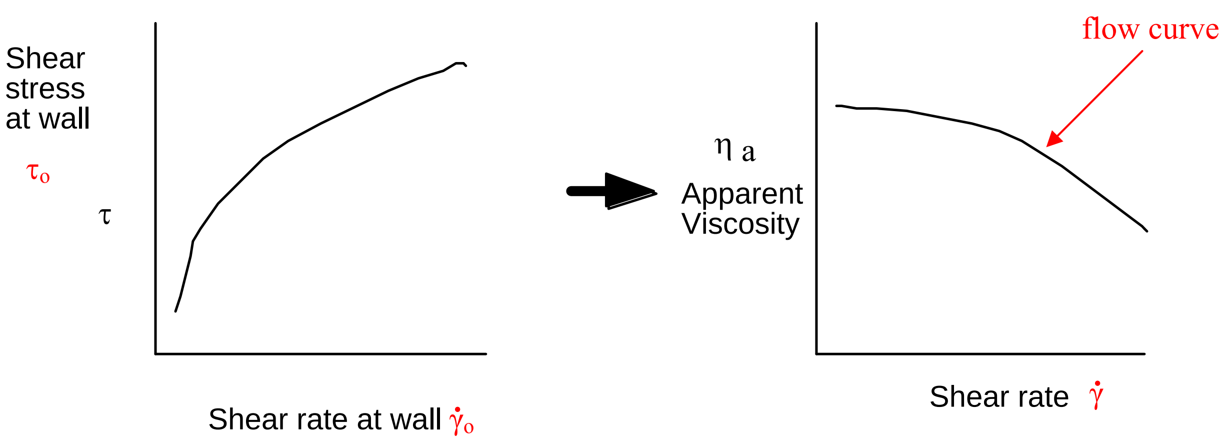
So if you know ΔP(Q)



From (1) yields τ_o, from (2) yield $\dot{\gamma}_o$

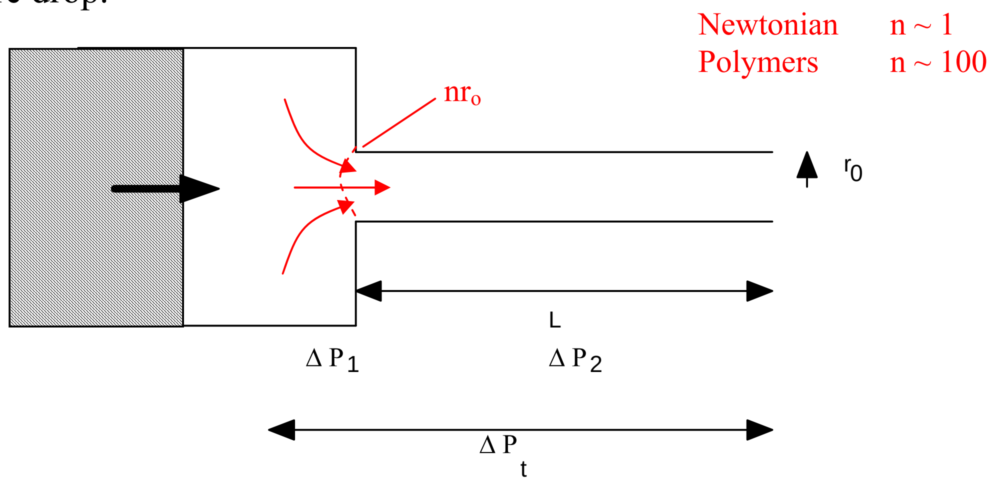
$\tau = \eta_a \dot{\gamma}$

$\tau_o = \eta_a \dot{\gamma}_o$



We have obtained the 'flow curve' for fluid without presupposing a rheology

A further complication. Less mathematical, but important. Entry pressure drop. Most capillary rheometers have an entry section of where there is an associated entry pressure drop.



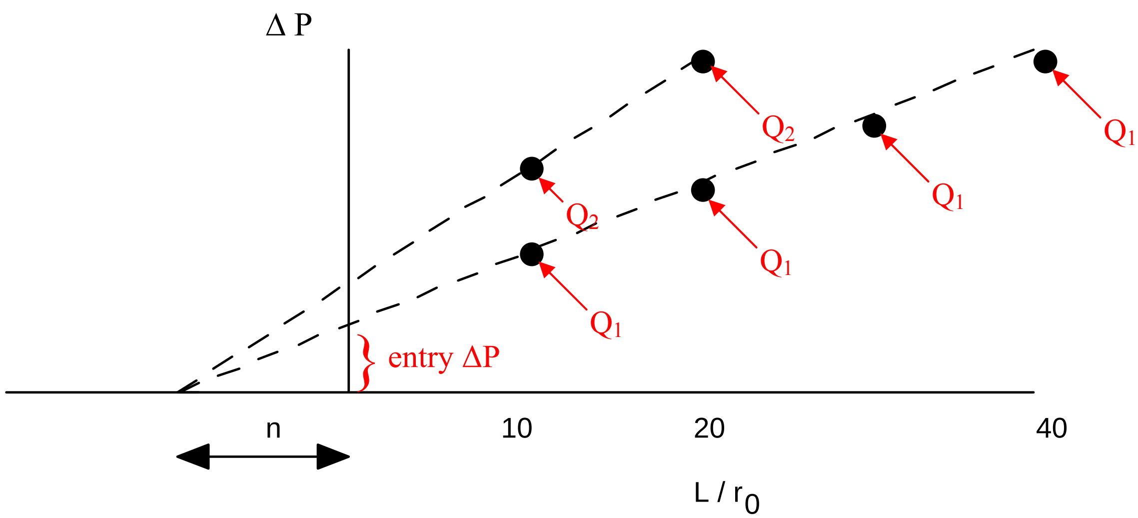
Use **"Bagley"** correction. Requires experiments with Different L/r_o ratios

Bagley (generally incorrectly) assumed that entry ΔP was equivalent to an added capillary length nr_o

$$\tau_o = \frac{\Delta P_t \, r_o}{2(L + nr_o)}$$

← added length

If Bagley assumption is true then the lines will go through a common intercept (sometimes they don't!)



Analytic and numerical solns give $\Delta P_1 = 2.3\ \tau_o$ $\qquad \tau_0$ in capillary

then n=1.15 But beware, for polymer fluids n >> 1

Rotational, (Torsion flow)

Relevant to Couette viscometers and stirred tank mixing vessels

Engineering flows

Beware Taylor Vortices (Large gaps > 1 cm) Re > 1

G.I. Taylor

Flow in circular orbits what is $\dot{\gamma}$?

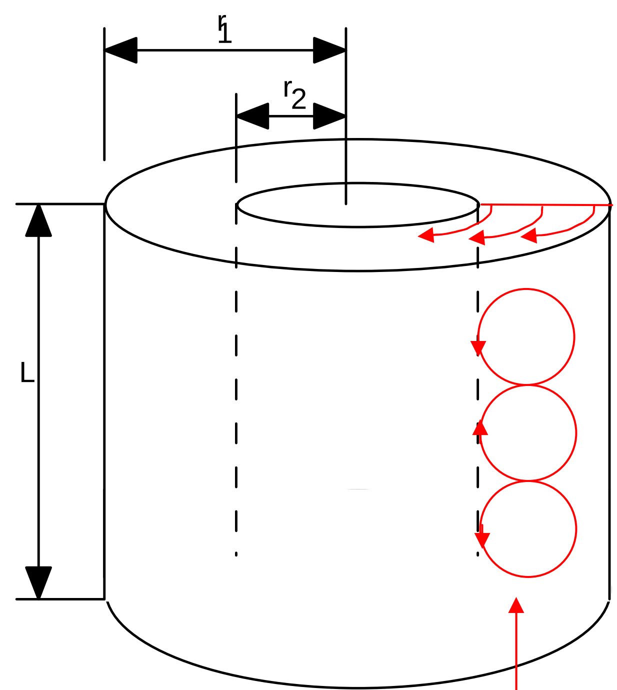
Couette:- Viscosity of gases $\eta \sim 10^{-5}$ Measured by Maurice Couette.

Concentric cylinder apparatus

Radius r_1 angular vel ω_1
Radius r_2 angular vel ω_2

$\omega = \frac{v}{r}$

What is $\dot{\gamma}$ for flows in concentric orbits?

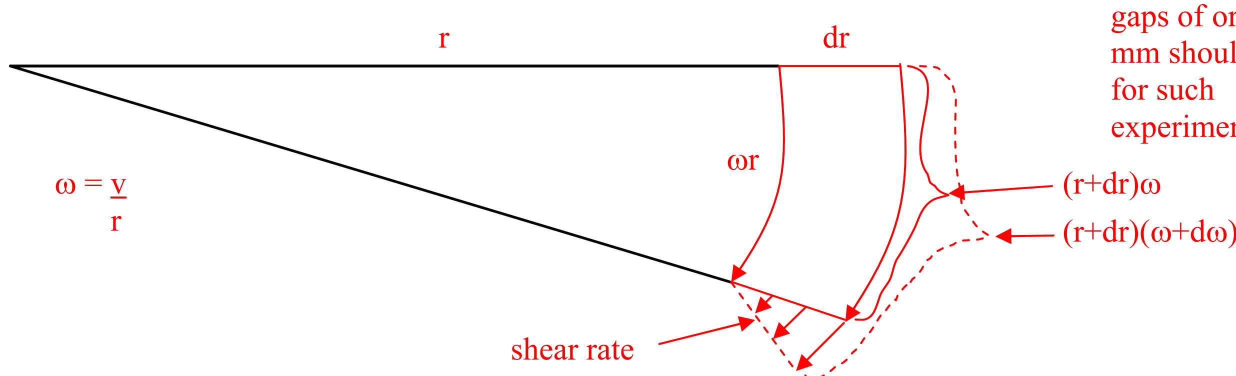


Toroidal vortices if Re > 1. These are caused by inertia and thus small gaps of order 1 mm should be used for such experiments

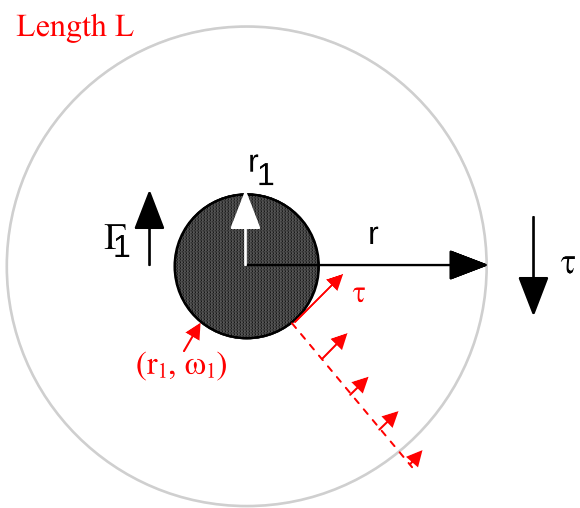
Remove rotational part of solid body rotation

$$\dot{\gamma} = \frac{(r + dr)(\omega + d\omega) - (r + dr)\omega}{dr} \qquad \dot{\gamma} = r\frac{d\omega}{dr}$$

$\omega = \frac{v}{r}$



Shaft rotating in a Newtonian fluid



Positive clockwise
Γ linear. Force balance
$P = m\ddot{x}$
equals 0

Net torque Γ_v given by

$$\Gamma_n = I\,\dot{\omega}$$

where I = moment of inertia

$\dot{\omega}$ = angular acceleration

For steady rotation

$$\dot{\omega} = 0$$

$$\therefore \Gamma_n = 0$$

$$\Gamma_n = \Gamma_1 + 2\Pi r^2 \tau L = 0$$

shaft — shear stress

$$\mathbf{\Gamma_1 = -2\Pi r^2 L\tau}$$

$$\tau = -\frac{\Gamma_1}{2\Pi L}\frac{1}{r^2}$$

Constitutive Equation $\tau = \eta\dot{\gamma} = \eta\, r\dfrac{d\omega}{dr}$ $\qquad \dot{\gamma} = r\dfrac{d\omega}{dr}$

So $\dfrac{\Gamma_1}{2\Pi L} = -\eta r^3 \dfrac{d\omega}{dr}$

BC $\omega = \omega_1$ at $r = r_1$

$\omega = 0$ at $r = r_2$ say

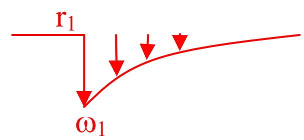
<u>at r</u> $\omega_1 - \omega(r) = \dfrac{\Gamma_1}{4\Pi\eta L}\left[\dfrac{1}{r_1^2} - \dfrac{1}{r^2}\right]$

Angular velocity at r

at r_1 gives   ω at r

$$\frac{\Gamma_1}{4\Pi\eta L} = \omega_1 \frac{r_1^2 r_2^2}{\left(r_2^2 - r_1^2\right)}$$

$$\Rightarrow \omega_1 - \omega = \omega_1 \frac{r_1^2 r_2^2}{\left(r_2^2 - r_1^2\right)}\left[\frac{1}{r_1^2} - \frac{1}{r_2^2}\right] \ldots (1)$$



Surprise result:- ω(r) ind of η

Velocity profile independent of viscosity. e.g. for water, bitumen and honey.

But torque will be different

$$\text{Shear rate } \dot{\gamma} = r\frac{d\omega}{dr} = -2\omega \frac{r_1^2 r_2^2}{\left(r_2^2 - r_1^2\right)} \frac{1}{r^2}$$

if $(r_2 - r_1)$ small and r_1 large

$\dot{\gamma} \approx$ a constant across gap

$$\text{So} \quad \eta = \frac{\Gamma_1}{4\Pi\omega L}\left[\frac{1}{r_1^2} - \frac{1}{r_2^2}\right]$$

The Couette viscometer

<u>Stirred tanks</u>   60 rpm $\omega_1 = 2\Pi = 6.28$ rad/s

Shaft   D = 25 mm

Eqn (1) gives $\omega = \dfrac{\omega_1}{2}$ at $r = 0.017$ m

Mixing very poor.
So use an impeller!

for small gap:-
Large r_1 = 20cm
Small gap $r_2 - r_1$ = 0.5mm ~ 1mm
Constant $\dot{\gamma}$:- across gap
Couette viscometer
Large surface area = large torque

Power Law Fluid in rotational flow

$\tau = k\dot{\gamma}^n$

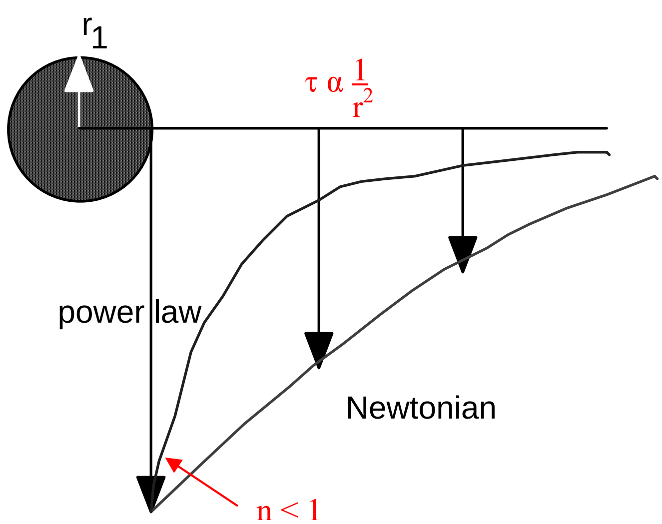
Torque equation
$$\Gamma_1 = -2\Pi\, r^2 L\, \tau$$
$$= -2\Pi\, r^2 Lk \left[r \frac{d\omega}{dr} \right]^n$$

As before n ≈ 0.6

$\gamma = r \frac{d\omega}{dr}$

Yields
$$\omega_1 - \omega(r) = \left[\frac{\Gamma_1}{2\Pi k L} \right]^{1/n} \frac{n}{2} \left[\frac{1}{r_1^{2/N}} - \frac{1}{r^{2/n}} \right]$$

For n < 1.

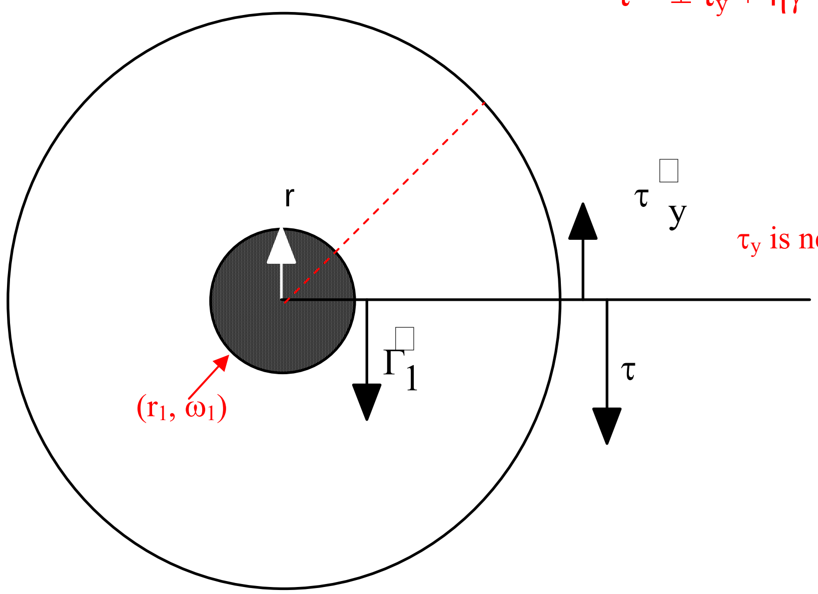


More localised than Newtonian. Power law mixing in rotational flow is more difficult than for Newtonian liquids.

Bingham Plastic in rotational flow

scalar and a property of the fluid

$$\tau = \pm\, \tau_y + \eta\dot{\gamma}$$



τ_y is negative

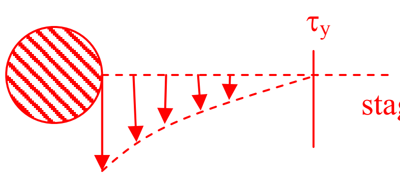
$$\Gamma_1 = -2\Pi r^2 L \tau \qquad \tau = -\frac{\Gamma_1}{2\Pi L}\frac{1}{r^2} \quad \text{as before}$$

$$\tau = -\tau_y + \eta\dot{\gamma}$$

$$\Gamma_1 = -2\pi L r^2\left[-\tau_y + \eta r\frac{d\omega}{dr}\right]$$

$$\omega_1 - \omega(r) = \frac{\tau_y}{\eta}\ln\frac{r_1}{r} + \frac{\Gamma_1}{4\Pi\eta L}\left[\frac{1}{r_1^2} - \frac{1}{r^2}\right]$$

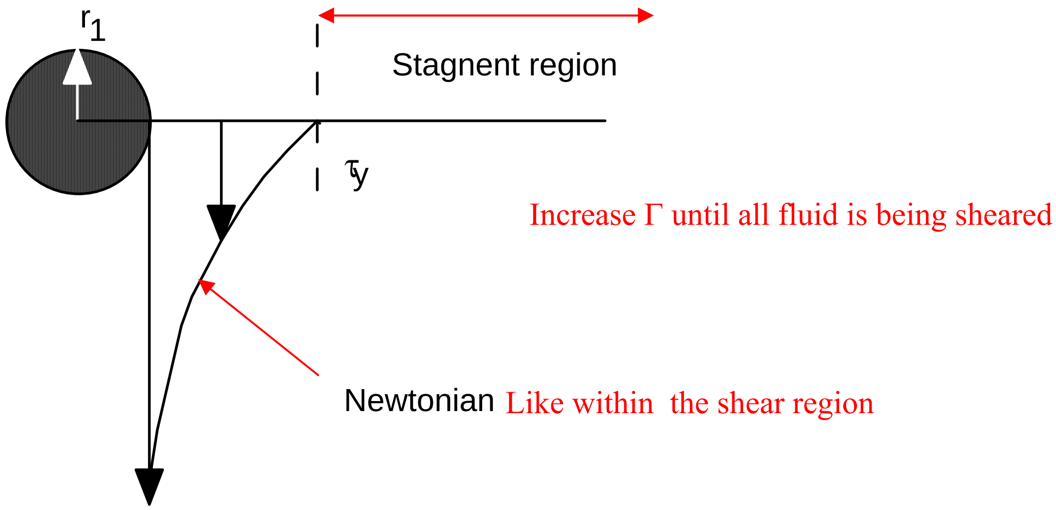
Binghams have a mixing problem



$r > r_2 \quad \dot{\gamma} = 0 \quad$ no flow

stagnant outer region

$$r_2^2 = \frac{\Gamma_1}{2\Pi L}\frac{1}{\tau_y}$$



Binghams can have serious mixing problems. Stirred vessels can have stagnant out regions.

Different geometries / Different forces balance elements.

Time for supervision 1