CET 2B Example Sheet 1 (The final one)

Rheology

Section 1 and 2.

1. Using a time dependant 'Cross' structure model. Determine the shear stress of a fluid that has been subjected to a constant shear rate $\dot{\gamma} = 5\ s^{-1}$ for a period of 5 s. Assume that the fluid started ***from rest*** with a viscosity of 3 Pas. Data for model given below.

If, after the initial 5 s at $\dot{\gamma} = 5\ s^{-1}$, the fluid is not sheared for a further 15 s, determine the fluids viscosity after this time. Following the 15 s no shear period, the fluid is sheared again at 20 s^{-1}, determine the shear stress after a period of 10 s at this shear rate. Draw sketches of the time evolution of the shear rate, viscosity and stress for the fluid subject to these boundary conditions.

Data: η_o = **3 N s/m^2**

η_∞ = **0.1 N s/m^2**

n = **0.6**

k_1 = **0.02 s^{-1}**

k_2 = **0.016 s^{-1}**

Answers:

τ_5 = **11.77 N/m^2**

η_{20} = **2.49 Pas**

τ_{30} = **19.28 N/m^2**

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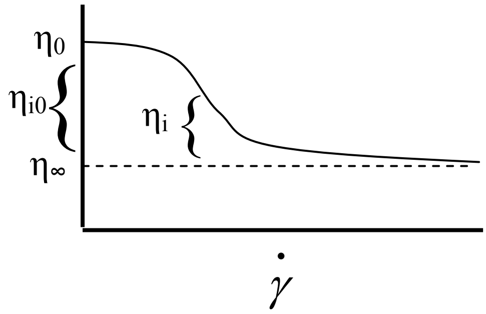
Region 1



$$\frac{dm}{dt} = -[k_2 + k_1\dot{\gamma}^n]m + k_2 m_0$$

$\frac{dm}{k_2 m_0 - \beta m} = dt,$ where $\beta = k_2 + k_1\dot{\gamma}^n$

Let $m = m_0$ at $t = 0$. Then

$$\frac{m}{m_0} = \frac{1}{\beta}[k_2 - (k_2 - \beta)\exp - \beta t]$$

Assume $\frac{\eta_i}{\eta_{i0}} = \frac{m}{m_0}$

$$\Rightarrow \eta_i = \frac{\eta_{i0}}{\beta}[k_2 - (k_2 - \beta)\exp(-\beta t)]$$

$$\eta_{i0} = \eta_0 - \eta_\infty$$

$$\eta_5 = \eta_i + \eta_\infty$$

$$\Rightarrow \eta_5 = \frac{\eta_0 - \eta_\infty}{\beta}[k_2 - (k_2 - \beta)\exp(-\beta t)] + \eta_\infty$$

$$\dot{\gamma} = 5 \Rightarrow \beta = 0.016 + 0.02 \times 5^{0.6} = 0.0685$$

$$\eta_5 = \frac{(3 - 0.1)}{0.0685}[0.016 - (0.016 - 0.0685)\exp(-0.0685 \times 5)] + 0.1 = 2.36\ Pa.s$$

$$\tau_5 = \eta_5 \times \dot{\gamma} = 2.36 \times 5 = 11.8\ N/m^2$$

Region 2

Shear stress is zero because $\dot{\gamma} = 0$ but viscosity will be changing with time:

$$\frac{dm}{dt'} = k_2\left[m_0 - m\right]t'$$

When $t' = 0$, let $m = m_5$ and $\eta_i = \eta_5$

$$\ln\frac{m_0 - m}{m_0 - m_5} = -k_2 t'$$

$$1 - \frac{m}{m_0} = \left(1 - \frac{m_5}{m_0}\right)\exp\left(-k_2 t'\right)$$

$$1 - \frac{\eta_i}{\eta_{io}} = \left(1 - \frac{\eta_{i5}}{\eta_{i0}}\right)\exp\left(-k_2 t'\right)$$

$$\eta_i = \eta_{i0}\left[1 - \left(1 - \frac{\eta_{i5}}{\eta_{i0}}\right)\exp\left(-k_2 t'\right)\right]$$

$$\eta_{i5} = \eta_5 - \eta_\infty$$

$$\eta_i = \left(\eta_0 - \eta_\infty\right)\left[1 - \left(1 - \frac{\eta_5 - \eta_\infty}{\eta_0 - \eta_\infty}\right)\exp\left(-k_2 t'\right)\right]$$

$t' = 15,\ \eta_{20} = 2.49\ Pa.s$

Region 3

$$\frac{dm}{dt''} = -\left[k_2 + k_1\dot{\gamma}^n\right]m + k_2 m_0$$

When $t'' = 0$, let $m = m_{20}$ and $\eta_i = \eta_{i20}$

$$-\frac{1}{\beta}\ln\left(\frac{k_2 m_0 - \beta m}{k_2 m_0 - \beta m_{20}}\right) = t''$$

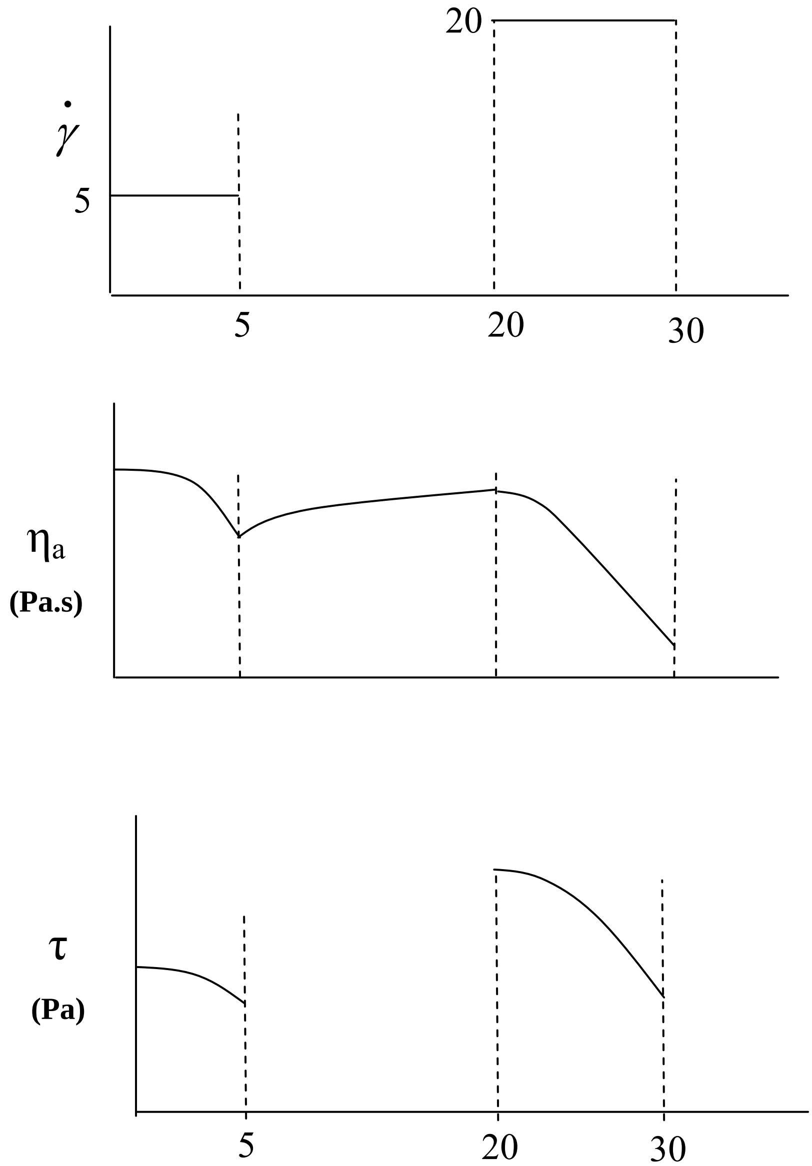
$$\frac{m}{m_0} = \frac{1}{\beta}\left[k_2 - \left(k_2 - \beta\frac{m_{20}}{m_0}\right)\exp\left(-\beta t''\right)\right]$$

$$\frac{\eta_i}{\eta_{i0}} = \frac{1}{\beta}\left[k_2 - \left(k_2 - \beta\frac{\eta_{i20}}{\eta_0}\right)\exp(-\beta t'')\right]$$

$$\eta_i = \frac{(\eta_0 - \eta_\infty)}{\beta}\left[k_2 - \left(k_2 - \beta\frac{\eta_{20} - \eta_\infty}{\eta_0 - \eta_\infty}\right)\exp(-\beta t'')\right]$$

$$t'' = 10,\ \eta_{30} = 0.963\ Pa.s$$

$$\tau_5 = \eta_{30} \times \dot{\gamma} = 2.36 \times 20 = 19.3\ N/m^2$$



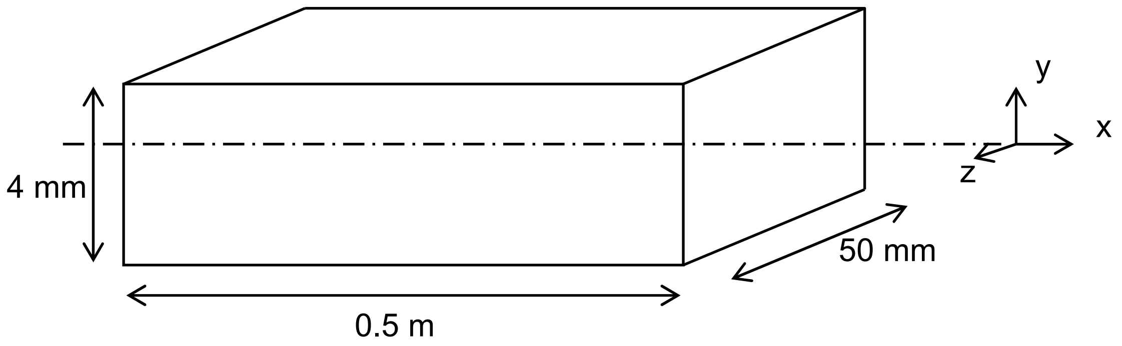
2. Fluid flows as a steady, two dimensional (x,y) laminar flow within a parallel walled rectangular section duct. The length of the duct x is 0.5 m, The duct width y, across which a transverse velocity profile develops, is 4 mm and the depth of the duct z is 50 mm. z=50>>y=4 so flow approximates to 2D.

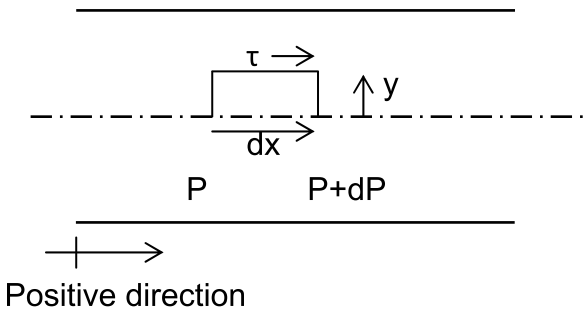
The fluid has been modelled in three ways.

(a)	Newtonian	η	=	1.3 N s/m^2
(b)	Power Law	k	=	40 N s$^{0.8}$/m^2
		n	=	0.2
(c)	Bingham	η	=	0.3 Ns/m^2
		τ_y	=	100 N/m^2

Derive expressions and sketch the velocity profiles for each of the above constitutive equation flow. Determine the volumetric flow rates predicted by each equation for a pressure drop of 0.3 bar along the length of the duct and determine the times for breakthrough (the time for the first part of an initially uniform pulse across the duct depth) to reach the end of duct) of a tracer and the mean residence time of the fluid in each case. Sketch the form of the residence time distribution that you would expect for each fluid.

Answers:	**Newtonian**	**1.23**	**x**	**10^{-5} m^3/s**
	Power Law	**1.39**	**x**	**10^{-5} m^3/s**
	Bingham	**2.08**	**x**	**10^{-6} m^3/s**





Force Balance:

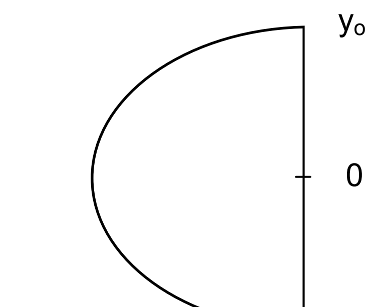
$$\tau.dx = dP.y$$

$$\Rightarrow \tau = \frac{dP}{dx}y = \frac{\Delta P}{L}y$$

Newtonian Fluid

$$\tau = \eta\frac{du}{dy} = \frac{\Delta P}{L}y$$

$$\eta.u = \frac{\Delta P}{L}.\frac{y^2}{2} + A$$



Boundary conditions:

$$u = 0, y = y_0 \Rightarrow A = -\frac{\Delta P}{L}.\frac{y_0^{\ 2}}{2}$$

$$\Rightarrow \quad u = -\frac{\Delta P}{2\eta L}\left[y_0^{\ 2} - y^2\right]$$

$$u_{\max} = -\frac{\Delta P}{2\eta L} y_0^{\ 2} = -\frac{0.3\times 10^5}{2\times 1.3\times 0.5}\times\left(2\times 10^{-3}\right)^2 = -0.092\ m/s$$

Breakthrough time, t_b:

$$t_b = \frac{L}{u_{\max}} = 5.42\ s$$

Volumetric flowrate, Q

$$Q = \int u.dA \Rightarrow Q = \frac{1}{2\eta}\left(\frac{\Delta P}{L}\right)\int_{-z_0}^{z_0}\int_{-y_0}^{y_0}\left(y_0^{\ 2} - y^2\right)dy.dz$$

$$= \frac{2z_0}{2\eta}\left(\frac{\Delta P}{L}\right)\int_{-y_0}^{y_0}\left(y_0^{\ 2} - y^2\right)dy$$

$$= \frac{z_0}{\eta}\left(\frac{\Delta P}{L}\right)\left(y_0^{\ 3} - \frac{y_0^{\ 3}}{3} - \left[-y_0^{\ 3} + \frac{y_0^{\ 3}}{3}\right]\right)$$

$$= \frac{4z_0}{3\eta}\left(\frac{\Delta P}{L}\right)y_0^{\ 3}$$

$$= \frac{4\times 25\times 10^{-3}\times 0.3\times 10^5\times\left(2\times 10^{-3}\right)^3}{3\times 1.3\times 0.5} = 1.23\times 10^{-5}\ m^3/s$$

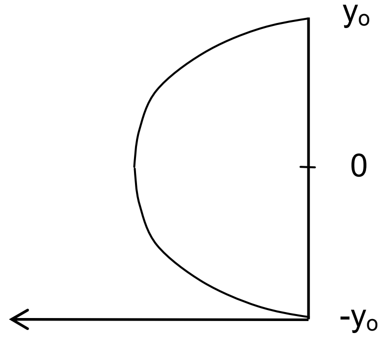
Mean residence time: $\bar{t} = \frac{V}{Q} = \frac{0.5\times 50\times 10^{-3}\times 4\times 10^{-3}}{1.23\times 10^{-5}} = 8.13\ s$

Power Law Fluid

$$\tau = k\left(\frac{du}{dy}\right)^n = y\frac{\Delta P}{L}$$

$$\left(\frac{\Delta P}{kL}\right)^{1/n} y^{(n+1)/n}\frac{n}{n+1} = u + B$$

Boundary conditions: $B = \left(\frac{\Delta P}{kL}\right)^{1/n} y_0^{(n+1)/n} \frac{n}{n+1}$



$$\Rightarrow u = -\left(\frac{\Delta P}{Lk}\right)^{1/n} \frac{n}{n+1}\left(y_0^{(n+1)/n} - y^{(n+1)/n}\right)$$

$$u_{\max} = -\left(\frac{\Delta P}{Lk}\right)^{1/n} \frac{n}{n+1} y_0^{(n+1)/n} = -0.081\, m/s$$

$$t_b = \frac{L}{u_{\max}} = 6.17 s$$

$$Q = \frac{n}{n+1}\left(\frac{\Delta P}{Lk}\right)^{1/n} \int_{-z_0}^{z_0}\int_{-y_0}^{y_0}\left(y_0^{(n+1)/n} - y^{(n+1)/n}\right) dy.dz$$

$$= \frac{2z_0 n}{n+1}\left(\frac{\Delta P}{Lk}\right)^{1/n} \int_{-y_0}^{y_0}\left(y_0^{(n+1)/n} - y^{(n+1)/n}\right) dy$$

$$= \frac{4z_0 n}{2n+1}\left(\frac{\Delta P}{Lk}\right)^{1/n} y_0^{(2n+1)/n}$$

$$= 1.39 \times 10^{-5}\ m^3/s$$

$$\bar{t} = \frac{V}{Q} = 7.20\ s$$

Bingham Fluid

$\tau = \tau_y + \eta \frac{du}{dy} = y\frac{\Delta P}{L}$ when $\tau > \tau_y$

$$\frac{\Delta P}{L}\frac{y^2}{2} - \tau_y y = \eta u + C$$

Boundary conditions: $C = \left(\frac{\Delta P}{L}\right)\frac{y_0^2}{2} - \tau_y y_0$

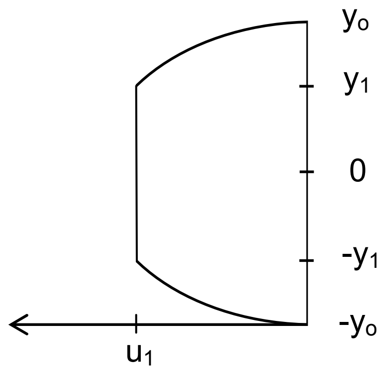
$$\Rightarrow u = \frac{1}{2\eta}\left(-\frac{\Delta P}{L}\right)\left(y_0^{\ 2} - y^2\right) + \frac{\tau_y}{n}\left(y_0 - y\right) \qquad \text{when} \qquad \tau > \tau_y$$

$$\tau < \tau_y \rightarrow \frac{du}{dy} = 0$$

$\tau = \tau_y$ when $y = y_1$

From the force balance:

$$\frac{\Delta P}{L} y_1 = \tau_y \Rightarrow y_1 = \frac{\tau_y}{\Delta P/L} = \frac{100}{\left(0.3\times 10^5\right)/0.5} = 1.67\ mm$$



$u = u_1$ when $y = y_1 \Rightarrow u_1 = -0.011\ m/s$

$$u_{\max} = u_1 = -0.011\ m/s$$

$$t_b = \frac{L}{u_{\max}} = 45.45\ s$$

Volumetric Flow Rate : $Q = Q_1 + Q_2$

$$y < y_1 \Rightarrow Q_1 = \int_{-z_0}^{z_0}\int_{-y_1}^{y_1} u_1\, dydz = 4 y_1 z_0 u_1 = 1.85\times 10^{-6}\ m^3/s$$

$y_0 < y < y_1$ and $-y_0 < y < -y_1$

$$\Rightarrow Q_2 = 2\int_{-z_0}^{z_0}\int_{-y_1}^{y_1}\left[\frac{1}{2\eta}\left(\frac{\Delta P}{L}\right)\left(y_0^{\ 2} - y^2\right) + \frac{\tau_y}{\eta}\left(y_0 - y\right)\right] dydz$$

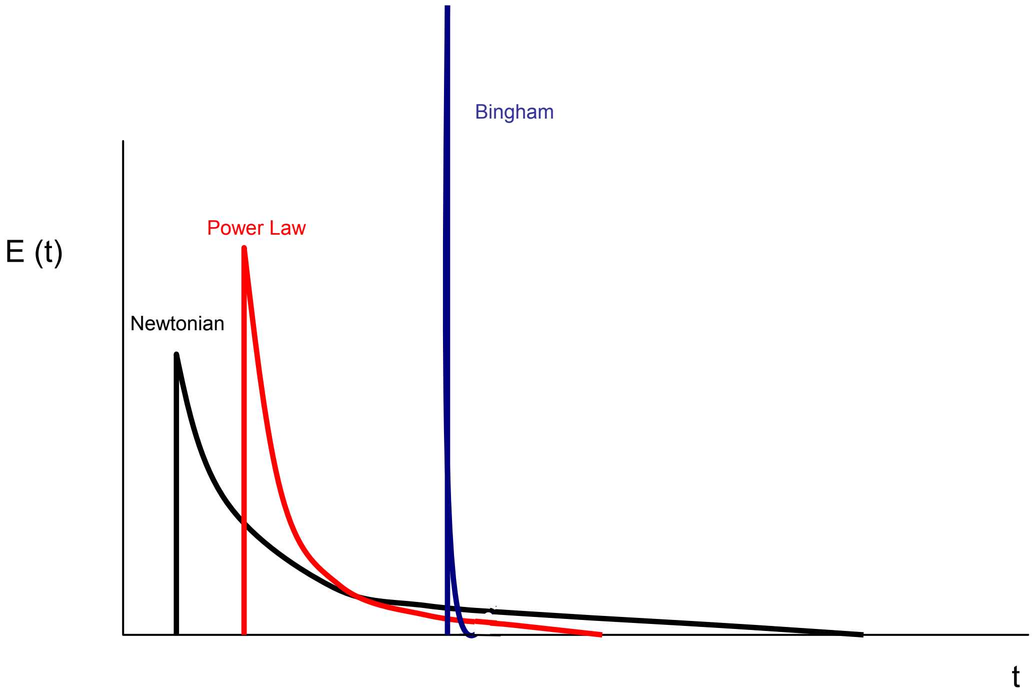
$$= \frac{4z_0}{\eta}\left[\frac{1}{2}\left(\frac{\Delta P}{L}\right)\left(\frac{2y_0^{\ 3}}{3} - y_0^{\ 2} y_1 + \frac{y_1^{\ 3}}{3}\right) + \tau_y\left(\frac{y_0^{\ 2}}{2} - y_0 y_1 + \frac{y_1^{\ 2}}{2}\right)\right]$$

$$= 2.47\times 10^{-7}\ m^3/s$$

$$Q = 1.85\times 10^{-6} + 2.47\times 10^{-7} = 2.10\times 10^{-6}\ m^3/s$$

$$\bar{t} = \frac{V}{Q} = 48\ s$$

Residence Time distributions:



3. Two fluids flow within a tube of radius r_o = 5 mm. In the central region of the tube r = 0 to r = 4 mm a Newtonian fluid, **A,** is flowing as a core of fluid. Within the annular region r = 4 mm to r_o = 5 mm, a Power Law fluid **B** is flowing in the same direction as fluid A.

Given a pressure gradient of, 5 x 10^4 Pa/m along the tube, determine,

a. The maximum velocity within the tube,

b. The velocity at the A/B interface

c. The volumetric flow rate.

Assume no slip boundary conditions at the tube walls and stress and velocity continuity across the A/B interface.

Data: Viscosity of fluid A. 10 Pas

Fluid B. $\tau = 10\,\dot{\gamma}^{0.5}$

Answers U_{max} = 0.147 m/s, $U_{A/B}$ = 0.127 m/s Q = 8.751 x 10^{-6} m^3/s

Force balance:

$$\Delta P \pi r^2 = \tau 2\pi r L \quad \Rightarrow \quad \tau = \frac{r\Delta P}{2L}$$

Let $X = \frac{\Delta P}{L}$

Fluid A: $$X\frac{r}{2} = \eta\frac{du}{dr} \quad \Rightarrow \quad u = \int \frac{X}{2\eta} r dr = \frac{X}{4\eta} r^2 + C$$

At $r = 0$, $$u = u_{\max} \quad \Rightarrow \quad u = u_{\max} - \frac{X}{4\eta} r^2$$

Fluid B: $$X\frac{r}{2} = K\left(\frac{du}{dr}\right)^{1/2} \quad \Rightarrow \quad u = \left[\frac{X}{2K}\right]^2 \int r^2$$

$$u = \left[\frac{X}{2K}\right]^2 \frac{r^3}{3} + C$$

$$u = \left[\frac{X}{2K}\right]^2 \frac{1}{3}\left(r_0^{\ 3} - r^3\right)$$

Match velocity at r_1

$$u_m = \left[\frac{X}{2K}\right]^2 \frac{1}{3}\left(r_0^{\ 3} - r_1^{\ 3}\right) + \frac{X}{4\eta} r_1^{\ 2}$$

Velocity at r_1 : $$u_1 = u_m - \frac{X}{4\eta} r_1^{\ 2}$$

Flow rates:

$$Q_1 = \int_0^{r_1} u 2\pi r dr = 2\pi \int \left(u_m r - \frac{X}{4\eta} r^3\right) dr = 2\pi \left[u_m \frac{r_1^{\ 2}}{2} - \frac{X}{16\eta} r_1^{\ 4}\right]$$

$$Q_2 = \int_{r_1}^{r_0} u 2\pi r dr = 2\pi \int \left[\frac{X}{2K}\right]^2 \frac{1}{3}\left(r_0^{\ 3} r - r^4\right) dr$$

$$= 2\pi\left[\frac{X}{2K}\right]^2 \frac{1}{3}\left[\frac{{r_0}^3 r^2}{2} - \frac{r^5}{5}\right]_{r_1}^{r_0}$$

$$Q = Q_1 + Q_2$$

Numbers:

$X = 5\times10^4$, $r_0 = 5\times10^{-3}$, $K = 10$, $r_1 = 4\times10^{-3}$ $\eta = 10$

$$u_m = \left[\frac{5\times10^4}{2\times10}\right]^2 \frac{1}{3}\left(5^3 - 4^3\right)10^{-9} + \frac{5\times10^4}{4\times10}\times4^2\times10^{-6}$$

$u_m = 0.147$ m/s

At 4mm, $u_1 = 0.147 - \frac{5\times10^4}{40}\left(4\times10^{-3}\right)^2 = 0.127$ m/s

$$Q_1 = 2\pi\left[0.147\left(\frac{4\times10^{-3}}{2}\right)^2 - \frac{5\times10^4\left(4\times10^{-3}\right)}{16\times10}\right] = 6.886\times10^{-6} \text{ m}^3\text{/s}$$

$$Q_2 = 2\pi\left[\frac{5\times10^4}{2\times10}\right]^2 \frac{1}{3}\left[\frac{\left(5\times10^{-3}\right)^3}{2}\left[5^2 - 4^2\right]\times10^{-6} - \left[5^5 - 4^5\right]\frac{\left(10^{-3}\right)^5}{5}\right]$$

$= 1.865\times10^{-6}$ m^3/s

$Q = 8.751\times10^{-6}$ m^3/s

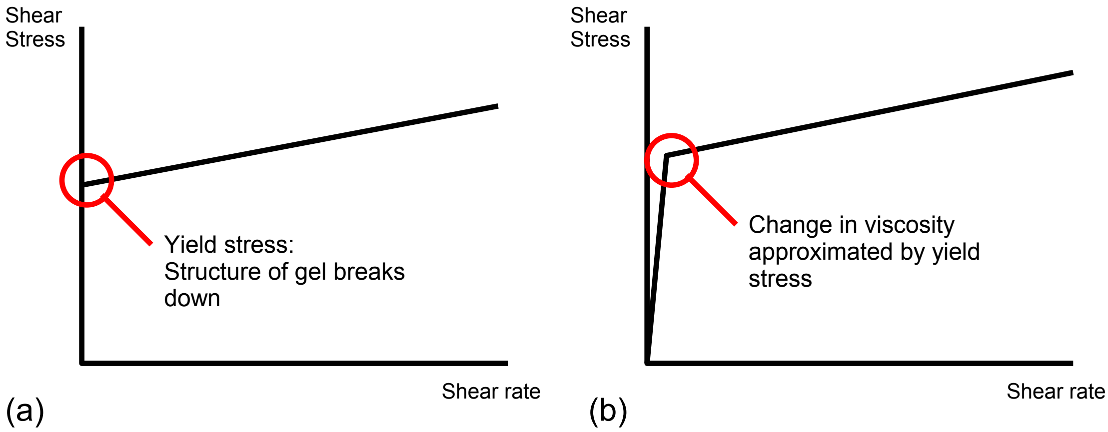
4. Write short notes on the following;

a) Why is water Newtonian, whereas a molten polyethylene is Non Newtonian?
The relaxation time, λ, of a material is the characteristic time scale for microstructure change in the fluid. If a material is sheared at rates ≥ 1/ λ then it will show non-Newtonian behaviour.
The relaxation time of a molecule is related to its size: larger molecules have a larger relaxation time.
Water is a small molecule, so it has a small relaxation time. You therefore need to shear it at extremely high shear rates in order to see non-Newtonian behaviour. These are not seen in engineering applications, and so water is treated as a Newtonian liquid.
Polyethylene is a long chain molecule and so it has a larger relaxation time. The shear rates required for non-Newtonian behaviour are likely to be seen in engineering applications, and therefore it is considered non-Newtonian.

b) Your blood shear thins; Why?
Blood is a concentrated suspension of red blood cells and other particles in water. There are interactions between the cells which are broken down as the sample is sheared. This causes it to shear thin (see Cross Equation analysis).

c) Molten polypropylene shear thins; Why?
Polypropylene consists of long chain molecules. As the sample is sheared, these chains are stretched, disentangled and orientated which causes the sample to shear thin.

d) Do fluids really have a yield stress as modeled by the Bingham equation?
Some fluids (e.g. gels) will genuinely have a yield stress as modeled by the Bingham equation (see (a) below). Other fluids will have a yield stress typified by a change in viscosity from a very high value to a much lower value (see (b) below).



e) Discuss the strength and weakness of modeling molten polymers using a power law equation and the strength and weakness of modeling tomato Ketchup using a Bingham equation.

Polymer using Power Law:

The power law is simple and easy to fit and use. It also can fit the shear thinning behaviour seen with polymers.

The power law does, not, however, fit the Newtonian plateau seen at low shear rates for molten polymers, but this will not be important at shear rates experienced in most engineering applications.

Ketchup using Bingham Equation:

The Bingham Equation is simple and easy to fit and use.

Ketchup may not have a yield stress as modeled by the Bingham equation, but this will not be important at shear rates experienced in most engineering applications.

The viscosity of the ketchup may not be independent of shear rate, in which case it would be better to use the Hershel-Buckley equation.

Note: neither the power law nor Bingham equations predict the viscoelastic behaviour seen for both these fluids.

f) Choose the most appropriate Non Newtonian constitutive equation for the following fluids and give reasons for your decision. Give estimates of either the zero shear rate viscosity or where appropriate the yield stress for each fluid.

i) Timotei shampoo

Shear thinning: Power Law/Cross, $\eta_0 \approx 1$ Pas

ii) Heinz Tomato Ketchup

Bingham, $\tau_y \approx 5$ Pa

iii) Potty putty

Bingham/Carreau, $\eta_0 \approx 10$ kPas

iv) Robinson College consume soup

Newtonian, $\eta_0 \approx 1$ mPas

v) North sea gas at 200bar, 20 centigrade

Newtonian, $\eta_0 \approx 0.1$ mPas

vi) Kings College thick English broth

Shear thinning: Power Law, $\eta_0 \approx 1$Pas

vii) Axial grease

Bingham, $\tau_y \approx 100$ Pa

viii) Lubricating oil

Newtonian, $\eta_0 \approx 1$Pas

ix) Honey

Newtonian, $\eta_0 \approx 1$Pas

x) Sainsbury yoghurt

Shear thinning: Power law/Cross, $\eta_0 \approx 10$Pas

xi) A Non Newtonian fluid of your own choice

xii) Your own blood

Cross, $\eta_0 \approx 0.1$Pas

xiii) Ink jet printer fluid

Newtonian (but some viscoelasticity at high shear rates), $\eta_0 < 20$ mPas

Section 3

5. Derive from first principles (without using your notes!) the constitutive equation for a 'Maxwell fluid' expressed in its

(a) Differential form
(b) Integral form, as a function of past strain rate
(c) Integral form, as a function of past strain

Using each of the above forms of equation, determine the response for both steady shear and stress relaxation after steady shear. Comment on the results.

What is the steady shear stress and apparent viscosity for a Maxwell fluid with the moduli specified below, at (a) $\dot{\gamma}$ = 10 s^{-1} and (b) $\dot{\gamma}$ = 100 s^{-1}. Comment on your result.

Answer **1.39 10^5 Pa, 1.39 10^6 Pa, 1.39 10^4 Pas**

What is the stress relaxation after steady shear for (a) $\dot{\gamma}$ = 10 s^{-1} and (b) $\dot{\gamma}$ = 100 s^{-1}, at t = 5 x 10^{-1} s and t = 5 s after shear cessation

Data:

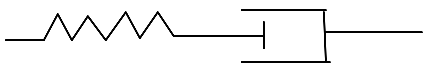
g_i	(N/m^2).	3 x 10^5,	5 x 10^4 ,	2 x 10^3,	1 x 10^3,
λ_i	(s).	10^{-3} ,	3.2 x 10^{-2} ,	1 ,	10,

Answers

At t=0.5s **1.07 10^5 Pa**
1.07 10^6 Pa

At t= 5s **6.08 10^4 Pa**
6.08 10^5 Pa

Maxwell Model:



$$\tau_1 = g\gamma_1 \qquad \tau_2 = \eta\dot{\gamma}_2$$

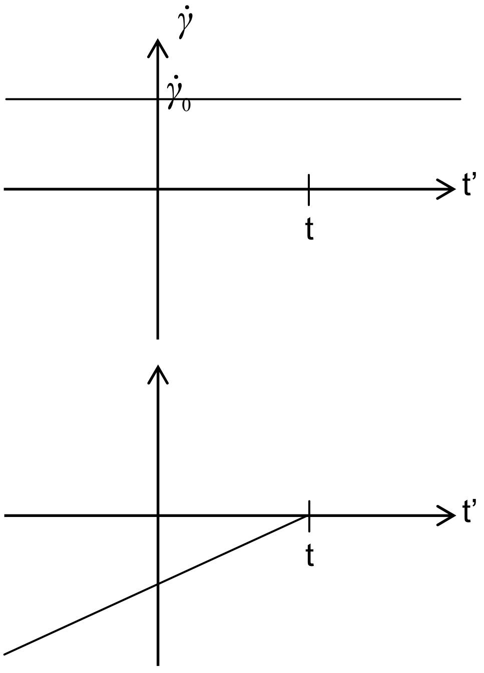
$$\gamma = \gamma_1 + \gamma_2 \qquad \tau_1 = \tau_2 = \tau$$

$$\dot{\gamma} = \dot{\gamma}_1 + \dot{\gamma}_2$$

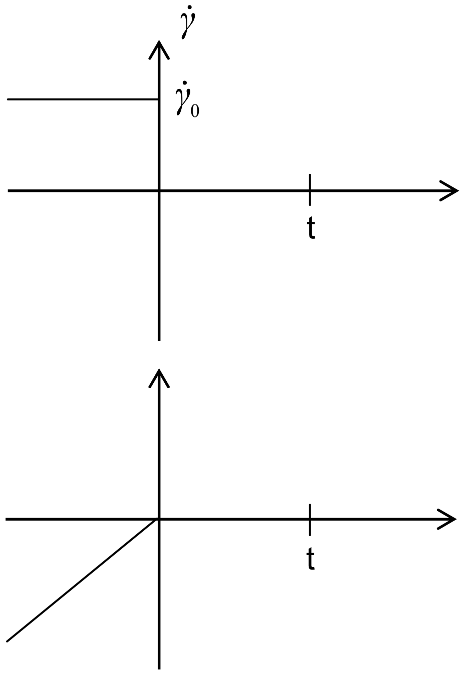
$$\dot{\gamma} = \frac{d\tau}{dt}\frac{1}{g} + \frac{\tau}{\eta}$$

$$g\dot{\gamma} = \frac{d\tau}{dt} + \frac{\tau}{\lambda}, \quad \lambda = \eta / g$$

Steady Shear



Stess Relaxation after Steady Shear



Differential form: $$g\dot{\gamma} = \frac{d\tau}{dt} + \frac{\tau}{\lambda}, \quad \lambda = \eta / g$$

Steady state: $$\frac{d\tau}{dt} = 0 \quad \Rightarrow \quad \tau = \eta\dot{\gamma}_0$$

Stress relaxation: $\dot{\gamma}=0 \quad \Rightarrow \quad \left[\frac{d\tau}{\tau}\right]_{\tau}^{\tau_0} = \left[\frac{dt}{\lambda}\right]_{t}^{0}$

$$\tau = \tau_0 e^{-t/\lambda}$$

Integral form based on past strain rate:

$$\tau(t) = \int_{-\infty}^{t} g e^{-(t-t')/\lambda} \dot{\gamma}(t')\, dt'$$

Steady shear: $\tau(t) = \dot{\gamma}_0 \int_{-\infty}^{t} g e^{-(t-t')/\lambda} dt' = \eta \dot{\gamma}_0$

Stress relaxation: $\tau(t) = \int_{-\infty}^{0} g e^{-(t-t')/\lambda} \dot{\gamma}_0 dt' + \underbrace{\int_{0}^{t} g e^{-(t-t')/\lambda} \underbrace{\dot{\gamma}}_{0} dt'}_{0} = \dot{\gamma}_0 \eta e^{-t/\lambda} = \tau_0 e^{-t/\lambda}$

Integral form based on the past strain:

$$\tau(t) = -\int_{-\infty}^{t} \frac{g}{\lambda} e^{-(t-t')/\lambda} \gamma(t,t')\, dt'$$

Steady shear: $\tau(t) = \dot{\gamma}_0 \int_{-\infty}^{t} \frac{g}{\lambda} e^{-(t-t')/\lambda} (t-t')\, dt' = \eta \dot{\gamma}_0$

Stress relaxation:

$$\tau(t) = \int_{-\infty}^{0} \frac{g}{\lambda} e^{-(t-t')/\lambda} \underbrace{\gamma(t,t')}_{-\dot{\gamma}(t-t')} dt' + \underbrace{\int_{0}^{t} \frac{g}{\lambda} e^{-(t-t')/\lambda} \underbrace{\gamma(t,t')}_{0} dt'}_{0} = \tau_0 e^{-t/\lambda}$$

$$\gamma(t,t') = \int_{t}^{t'} \dot{\gamma}(t'')\, dt'' = \begin{cases} -\dot{\gamma}(t-t') & for \quad -\infty < t' \le 0 \\ 0 & for \quad 0 < t' \le t \end{cases}$$

Numerical Part

NB : Spectrum of relaxation times $\rightarrow \tau = \sum_i \tau_i$

Steady shear:

$\tau = \sum_i g_i \lambda_i \dot{\gamma}_0$

$$\dot{\gamma}_0 = 10/s \Rightarrow \tau = 1.39 \times 10^5 \ Pa$$
$$\dot{\gamma}_0 = 100/s \Rightarrow \tau = 1.39 \times 10^6 \ Pa$$

$$\eta_a = \frac{\tau}{\dot{\gamma}} = 1.39 \times 10^4 \ Pa.s$$

Stress relaxation after steady shear:

$$\tau = \dot{\gamma}_0 \sum_i g_i \lambda_i \exp\left(\frac{-t}{\lambda_i}\right)$$

$$\dot{\gamma}_0 = 10/s, t = 0.5 \ s \Rightarrow \tau = 1.07 \times 10^5 \ Pa$$
$$\dot{\gamma}_0 = 100/s, t = 0.5 \ s \Rightarrow \tau = 1.07 \times 10^6 \ Pa$$
$$\dot{\gamma}_0 = 10/s, t = 5 \ s \Rightarrow \tau = 6.08 \times 10^4 \ Pa$$
$$\dot{\gamma}_0 = 100/s, t = 5 \ s \Rightarrow \tau = 6.08 \times 10^5 \ Pa$$

6. A viscoelastic fluid obeys the following constitutive equation

$$\tau(t) = -\int_{-\infty}^{t} \frac{g}{\lambda} e^{-(t-t')/\lambda} \gamma(tt')dt'$$

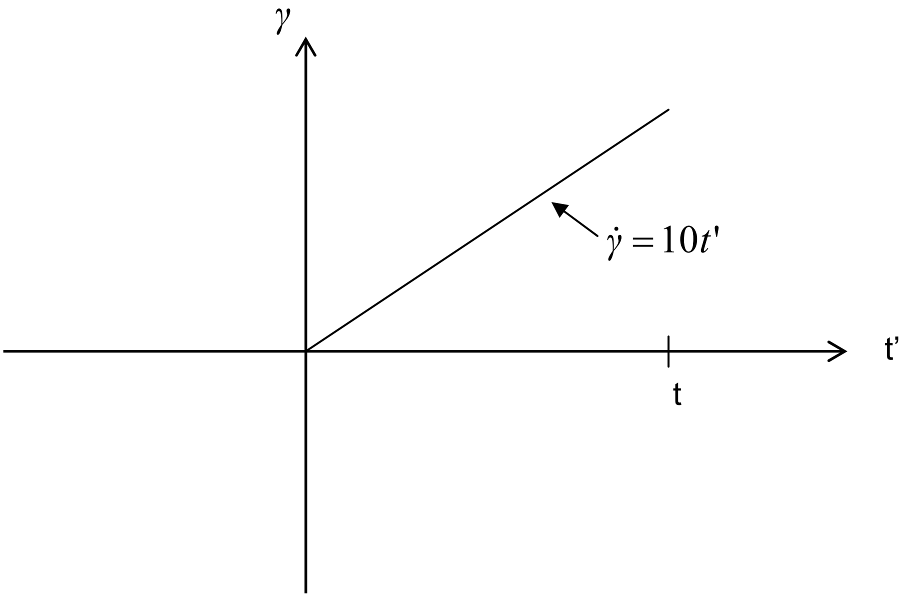
where g = 10^3 Pa and λ = 2 s.

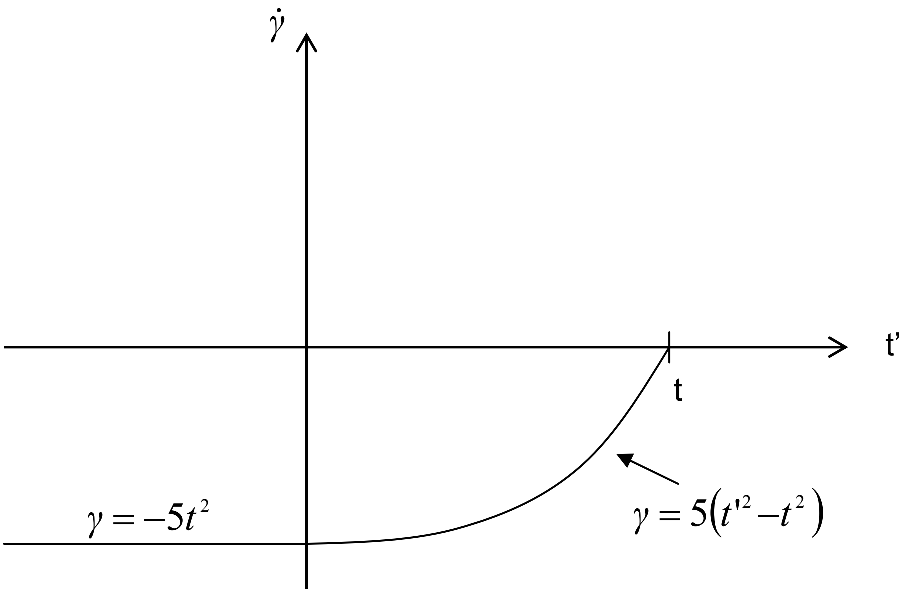
The fluid starts from a zero stress at time t'= 0 and is then subjected to a strain rate $\dot{\gamma}$ given by

$$\dot{\gamma} = 10\,t', \quad s^{-1}$$

where t' is the time that has evolved from the start time t'=o.
Determine the stress after a time t = 6 s.
Answer: (big calc!) τ = 82.0 **kPa**

Solution:





$$\gamma(t,t') = \int_t^{t'} \dot{\gamma}(t'')dt'' = \begin{cases} \int_t^{t'} 10t''.dt'' = 5\left(t'^2 - t^2\right) & 0 < t' < t \\ \int_t^{0} 10t''.dt'' + \int_0^{t'} 0.dt'' = -5t'^2 & -\infty < t' < 0 \end{cases}$$

$$\tau(t) = -\int_{-\infty}^{t} \frac{g}{\lambda} e^{-(t-t')/\lambda} \gamma(t,t')\, dt'$$

$$= -\int_{-\infty}^{0} \frac{g}{\lambda} \exp\left(\frac{-(t-t')}{\lambda}\right).-5t^2.dt' - \int \frac{g}{\lambda} \exp\left(\frac{-(t-t')}{\lambda}\right).5\left(t'^2 - t^2\right).dt'$$

Consider $\int_{-\infty}^{0} \frac{g}{\lambda} \exp\left(\frac{-(t-t')}{\lambda}\right).5t^2.dt'$

Let $a = t - t' \Rightarrow da = -dt'$

$$-\int_{\infty}^{t} 5t^2 \frac{g}{\lambda} \exp\left(\frac{-a}{\lambda}\right).da = \left[5t^2 g \exp\left(\frac{-a}{\lambda}\right)\right]_{\infty}^{t} = 5t^2 g \exp\left(\frac{-t}{\lambda}\right)$$

Consider $-\int \frac{g}{\lambda} \exp\left(\frac{-(t-t')}{\lambda}\right).5\left(t'^2 - t^2\right).dt'$

$$= \left[-5g\left(t'^2 - t^2\right)\exp\left(\frac{-(t-t')}{\lambda}\right)\right]_0^t + \int_0^t 10gt' \exp\left(\frac{-(t-t')}{\lambda}\right).dt'$$

$$= -5gt^2 \exp\left(\frac{-t}{\lambda}\right) + \left[10g\lambda t' \exp\left(\frac{-(t-t')}{\lambda}\right)\right]_0^t - \int_0^t 10g\lambda \exp\left(\frac{-(t-t')}{\lambda}\right).dt'$$

$$= -5gt^2 \exp\left(\frac{-t}{\lambda}\right) + 10g\lambda t - 10g\lambda^2 + 10g\lambda^2 \exp\left(\frac{-t}{\lambda}\right)$$

$$\tau = 10g\lambda t - 10g\lambda^2 + 10g\lambda^2 \exp\left(\frac{-t}{\lambda}\right)$$

$$= 10g\lambda\left(t - \lambda + \lambda \exp\left(\frac{-t}{\lambda}\right)\right)$$

$$= 10 \times 10^3 \times 2 \times \left(6 - 2 + 2\exp\left(\frac{-6}{2}\right)\right) = 82.0\ kPa$$

7. Write short notes on the following

i) All I need to know is the differential Maxwell equation, there is no difference in formulating the integral form, its more complex and a waste of time.
The integral Maxwell is derived from the differential form, so it should give the same result. However, people have improved the model by including additional factors in either the differential or integral forms. For example, the Wagner damping factor was devised for the integral form and so cannot be easily applied to the differential form.

ii) The integral Maxwell equation is the answer to all non linear viscoelastic problems
In theory, the integral Maxwell with Wagner damping factor could fit any data, so long as the spectrum of relaxation times is large enough. However, because this form of the model relies on past time, and you always have to work backwards from the point you are interested in, it requires an enormous amount of computing power to model complex flows.

iii) Why are the viscoelastic properties of water and low viscosity fluids not particularly important in terms of most engineering problems? Why is the viscoelasticity of printing inks important even though the zero shear viscosity is low?
Water and low viscosity fluids will be made up of small (low molecular weight) molecules. Therefore, they will have short relaxation times.
The relaxation time is the characteristic time scale for microstructure change in the fluid as so viscoelastic behaviour is seen at shear rates $\geq 1/\lambda$.
Therefore, even if a fluid has a short relaxation time, it will show viscoelastic behaviour at sufficiently high shear rates. The majority of engineering applications do not require such high shear rates and so the viscoelastic properties of water and low viscosity fluids are not particularly important.
In inkjet printers, very high shear rates are generated, and so the viscoelasticity of printing inks becomes significant even though their zero shear viscosities are low.

iv) You are applying for a job with Sainsburys; explain to the non technical interviewer why Sainsburys need to have in house expertise on viscoelastic properties.

The quality of various liquid products can be assessed by looking at their rheology. Sainsbury's therefore need to employ a rheology specialist for quality control of their goods, particularly natural products bought directly from farms (manufactured products like shampoo are likely to be tested by their manufacturer).

Section 4 of Example Sheet

8. A stress matrix, in a laboratory co-ordinate frame, for a flowing polymer is given below in MPa.

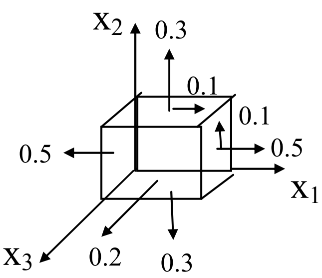
$$\begin{vmatrix} \mathbf{0.5} & \mathbf{0.1} & \mathbf{0} \\ \mathbf{0.1} & \mathbf{0.3} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0.2} \end{vmatrix}$$

Consider a new co-ordinate frame where the x_1 axis is rotated 30° anticlockwise in the plane x_1, x_2.

(i) Determine the direction cosine matrix between the new and old laboratory frame.
(ii) Determine the full stress matrix in the new co-ordinate frame.
(iii) Show that both the first and second invariants of the stress matrix are the same for both co-ordinate frames.

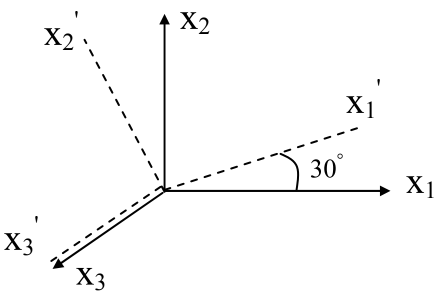
Ans $I_1 = 1$, $I_2 = 0.30$

Old Coordinate frame:



$$\begin{bmatrix} 0.5 & 0.1 & 0 \\ 0.1 & 0.3 & 0 \\ 0 & 0 & 0.2 \end{bmatrix}$$

Coordinate rotation:



$$\begin{bmatrix} \cos 30 & \cos 60 & 0 \\ \cos 120 & \cos 30 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0.866 & 0.5 & 0 \\ -0.5 & 0.866 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Stress matrix in the new coordinate frame:

$$\sigma'_{ij} = a_{ik} a_{jl} \sigma_{kl}$$

We have five non-zero σ_{kl} coefficients.

$$\sigma'_{ij} = a_{i1} a_{j1} \sigma_{11} + a_{i1} a_{j2} \sigma_{12} + a_{i2} a_{j1} \sigma_{21} + a_{i2} a_{j2} \sigma_{22} + a_{i3} a_{j3} \sigma_{33}$$

So

$$\sigma'_{ij} = a_{i1} a_{j1} 0.5 + a_{i1} a_{j2} 0.1 + a_{i2} a_{j1} 0.1 + a_{i2} a_{j2} 0.3 + a_{i3} a_{j3} 0.2$$

We have, in principle, five stress coefficients to compute in new coordinate frame.

$$\sigma'_{11} = a_{11} a_{11} 0.5 + a_{11} a_{12} 0.1 + a_{12} a_{11} 0.1 + a_{12} a_{12} 0.3 + a_{13} a_{13} 0.2$$
$$= (0.866)^2 0.5 + 0.866(0.5)0.1 + 0.5(0.866)0.1 + (0.5)^2 0.3 = 0.5366$$

$$\sigma'_{12} = a_{11} a_{21} 0.5 + a_{11} a_{22} 0.1 + a_{12} a_{21} 0.1 + a_{12} a_{22} 0.3 + a_{13} a_{23} 0.2 = -0.0366$$

Lets do σ'_{21} (need to be the same as σ'_{12} as stress tensor is symmetric). This is a useful check to see if sums are correct.

$$\sigma'_{21} = a_{21} a_{11} 0.5 + a_{21} a_{12} 0.1 + a_{22} a_{11} 0.1 + a_{22} a_{12} 0.3 + a_{23} a_{13} 0.2 = -0.0366$$

$$\sigma'_{22} = a_{21}a_{21}0.5 + a_{21}a_{22}0.1 + a_{22}a_{21}0.1 + a_{22}a_{22}0.3 + a_{23}a_{23}0.2 = 0.2634$$

$$\sigma'_{31} = 0$$
$$\sigma'_{23} = 0$$
$$\sigma'_{33} = 0.2$$

New stress matrix:

$$\begin{bmatrix} 0.5366 & -0.0366 & 0 \\ -0.0366 & 0.2634 & 0 \\ 0 & 0 & 0.2 \end{bmatrix}$$

Stress invariants

$$I_1 = \sigma_{11} + \sigma_{22} + \sigma_{33}$$

$$\left.\begin{aligned} I_1(old) &= 0.5 + 0.3 + 0.2 = 1.0 \\ I_1(new) &= 0.5366 + 0.2634 + 0.2 = 1.0 \end{aligned}\right\} I_1(old) = I_1(new)$$

$$I_2 = \sigma_{11}\sigma_{22} + \sigma_{11}\sigma_{33} + \sigma_{22}\sigma_{33} - \sigma_{12}^2 - \sigma_{23}^2 - \sigma_{13}^2$$

$$\left.\begin{aligned} I_2(old) &= 0.5(0.3) + 0.5(0.2) + 0.3(0.2) - (0.1)^2 = 0.3 \\ I_2(new) &= 0.5366(0.2634) + 0.5366(0.2) + 0.2634(0.2) - (0.0366)^2 = 0.3026 \end{aligned}\right\} I_2(old) = I_2(new)$$

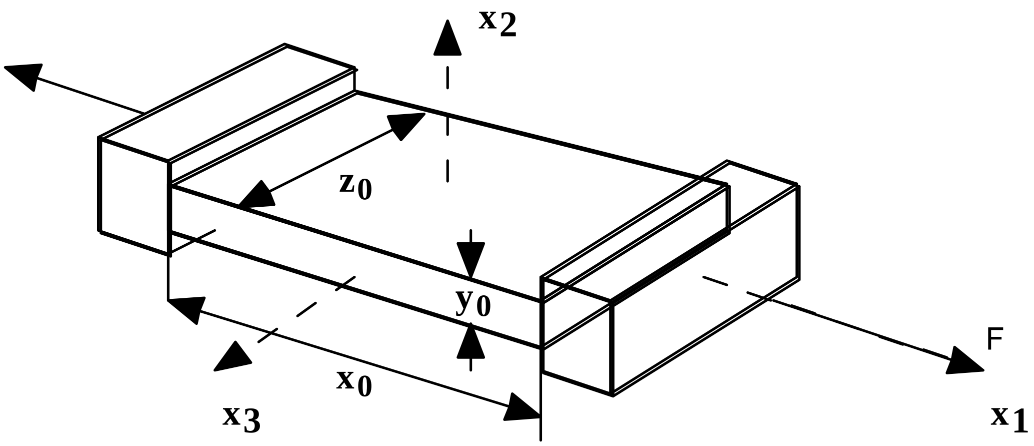
Magnitude of First and second stress invariants are the same for different coordinate frames.

9.

(a) Derive a relation between the extensional strain rates $\dot{\varepsilon}_{11}, \dot{\varepsilon}_{22}, \dot{\varepsilon}_{33}$ for an incompressible fluid.

(b) Write down the strain rate matrix for a 2D pure shear rate deformation.

(c) Using the generalised definition of Newtonion viscosity, derive a relation for extensional viscosity in pure shear.



(d) A strip of molten polymer having initial dimensions $x_o = 3$ mm $y_o = 1$ mm $z_0 = 5$ mm is held in a device shown in the diagram. The polymer is then stretched in the x_1 direction and data relating to the extension and tensile force F is given below.

(i) show that the deformation corresponds to a constant extension rate $\dot{\varepsilon}_{11}$ and determine the magnitude of $\dot{\varepsilon}_{11}$.

(ii) Show that the data is consistent with the polymer behaving as a Newtonian fluid and determine both the extensional and shear viscosity of the polymer. Assume that the z_0 dimension does not change during deformation and that the polymer is incompressible.

Time	s	0.5	1.0	1.5
Position of moving end	mm	4.9	8.15	13.4
Force N	N	$6.1\ 10^{-2}$	$3.7\ 10^{-2}$	$2.2\ 10^{-2}$

a)

$$\dot{\varepsilon}_{11} = \frac{\partial u}{\partial x_1} \qquad \dot{\varepsilon}_{22} = \frac{\partial v}{\partial x_2} \qquad \dot{\varepsilon}_{33} = \frac{\partial w}{\partial x_3}$$

The net outflow in x_1 direction:

$$\left(\rho u + \frac{\partial \rho u}{\partial x_1} dx_1\right) dx_2 dx_3 - \rho u dx_2 dx_3 = \frac{\partial \rho u}{\partial x_1} dx_1 dx_2 dx_3$$

$$\left(\rho v + \frac{\partial \rho v}{\partial x_2} dx_2\right) dx_1 dx_3 - \rho v dx_1 dx_3 = \frac{\partial \rho v}{\partial x_2} dx_1 dx_2 dx_3$$

$$\left(\rho w + \frac{\partial \rho w}{\partial x_3} dx_3\right) dx_1 dx_2 - \rho w dx_1 dx_2 = \frac{\partial \rho w}{\partial x_3} dx_1 dx_2 dx_3$$

The net outflow=accumulation

$$\frac{\partial \rho u}{\partial x_1} dx_1 dx_2 dx_3 + \frac{\partial \rho v}{\partial x_2} dx_1 dx_2 dx_3 + \frac{\partial \rho w}{\partial x_3} dx_1 dx_2 dx_3 = \frac{\partial \rho}{\partial t} dx_1 dx_2 dx_3$$

For an incompressible fluid $\frac{\partial \rho}{\partial t} = 0$ therefore:

$$\frac{\partial u}{\partial x_1} + \frac{\partial v}{\partial x_2} + \frac{\partial w}{\partial x_3} = 0 \quad \Rightarrow \quad \dot{\varepsilon}_{11} + \dot{\varepsilon}_{22} + \dot{\varepsilon}_{33} = 0$$

b)

$$\begin{bmatrix} \dot{\varepsilon}_{11} & 0 & 0 \\ 0 & \dot{\varepsilon}_{22} & 0 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} \dot{\varepsilon} & 0 & 0 \\ 0 & -\dot{\varepsilon} & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\dot{\varepsilon}_{11} + \dot{\varepsilon}_{22} + 0 = 0 \quad \dot{\varepsilon}_{11} = -\dot{\varepsilon}_{22} \quad \dot{\varepsilon}_{11} = \dot{\varepsilon}$$

c)

$$\sigma_{ii} = -P + 2\eta\dot{\varepsilon}_{ii}$$

$$\sigma_{11} = -P + 2\eta\dot{\varepsilon}$$

$$\sigma_{22} = -P - 2\eta\dot{\varepsilon}$$

$$\frac{\sigma_{11} - \sigma_{22}}{\dot{\varepsilon}} = \eta_c = 4\eta$$

d)

i) $\dot{\varepsilon} = \frac{d\varepsilon}{dt} = \frac{1}{l}\frac{dl}{dt}$

if $\dot{\varepsilon} = cte \quad \Rightarrow \quad \dot{\varepsilon}\int_0^t dt = \int_{l_0}^{l} \frac{dl}{l}$

$$\dot{\varepsilon}t = \ln\frac{l}{l_0} \quad \Rightarrow \quad \dot{\varepsilon} = \frac{1}{t}\ln\frac{l}{l_0}$$

Time	S	0.5	1.0	1.5
Length	mm	4.9	8.15	13.4
Strain rate	1/s	0.98	0.99	0.99

So $\dot{\varepsilon} \approx 1$ and is a constant value.

ii) $\eta_e = \frac{\sigma_{11} - \sigma_{22}}{\dot{\varepsilon}} = \frac{\sigma_{11} - 0}{\dot{\varepsilon}} = \frac{Force}{Area}\frac{1}{\dot{\varepsilon}}$

Incompressible fluid: $x_0 y_0 z_0 = xyz_0 \Rightarrow y = \frac{x_0 y_0}{x}$

Time	S	0.5	1.0	1.5
Length	mm	4.9	8.15	13.4
Strain rate	1/s	0.98	0.99	0.99
Force N	N	$6.1\ 10^{-2}$	$3.7\ 10^{-2}$	$2.2\ 10^{-2}$
Stress	Pa	$2.0\ 10^{4}$	$2.0\ 10^{4}$	$2.0\ 10^{4}$

$$\eta_e = 2.0 \times 10^4\, Pa.s$$

$$\eta_{ss} = \frac{2.0 \times 10^4}{4} = 5 \times 10^3\, Pa.s$$

10. A narrow gap Couette apparatus is used to measure the flow properties of a polymer solution. Using optical techniques it was possible to establish, for a certain flow condition, that the direction of the Principal axis was at an angle of 15^0 to the direction of flow, in the plane of shear of the flow. The magnitude of the Principal Stress Difference was 30 kPa.

a) If the Couette gap is 1 mm, the inner rotor diameter 30 mm, the depth of fluid 80 mm; determine the torque that would be recorded on the rotor inner cylinder. Use tensor transformation to calculate the shear stress.

b) If the outer cylinder is static and the inner cylinder rotates at an angular velocity of 300 $rads^{-1}$, determine,

i) The apparent shear viscosity of the fluid.

ii) The Normal Stress Difference of the fluid.

In the “old” coordinate frame:

Stress matrix (principal stress) $=\begin{bmatrix} \sigma_{11} & 0 & 0 \\ 0 & \sigma_{22} & 0 \\ 0 & 0 & 0 \end{bmatrix}$

Direction cosine matrix = $\begin{bmatrix} \cos 15 & \cos 75 & 0 \\ \cos 105 & \cos 15 & 0 \\ 0 & 0 & 1 \end{bmatrix} \Rightarrow \begin{bmatrix} 0.966 & 0.2588 & 0 \\ -0.2588 & 0.9660 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

In the new “laboratory” coordinate frame:

$$\sigma'_{ij} = a_{ik} a_{jl} \sigma_{kl}$$

$$\begin{aligned}
\sigma_{12} = \tau_{12} &= a_{11} a_{21} \sigma_{11} + a_{12} a_{22} \sigma_{22} \\
&= (-0.966)(0.2588)\sigma_{11} + (0.2588)(0.966)\sigma_{22} \\
&= -0.25\sigma_{11} + 0.25\sigma_{22} \\
\tau_{12} &= -0.25(\sigma_{11} - \sigma_{22}) = -0.25(30\times10^{3}) = -7.5\times10^{3}
\end{aligned}$$

$$|Torque| = \tau 2\pi r L \times r = 2\pi\tau L r^2$$

$$= 2\pi(7.5\times10^{3})80\times10^{-3}(15\times10^{-3})^2 = 0.8482$$

Narrow Gap: $\dot{\gamma} = -\dfrac{V}{\delta}$

$$|\dot{\gamma}| = \frac{300\times15\times10^{-3}}{1\times10^{-3}} = 4.5\times10^{3}\ s^{-1}$$

11. Write brief notes to test your broader understanding of rheology and processing.

a) Describe the useful rheological features of six different complex fluids that occur either during processing or as a product. In each case identify the most appropriate model to describe the fluids properties.

b) Identify three situations where modeling a complex fluid flowing within a complex flow would/could add value to a Company process.

c) Describe why tensorial aspects of Stress, strain and Strain rate need to be considered when considering **both** complex fluids and complex flows.